1. INTRODUCTION

Functions of Two Variables

One of the main differences between functions of one variable and functions of two variables is that functions of one variable are considered in an interval on the number line, whereas functions of two variables are considered in an open disc in the xy-plane. That is, with a function of one variable, $|x - a| < \delta$ means that within an interval, the distance of x to a is always less than δ. With a function of two variables, $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ means that the point (a, b) lies within an open circle (disc) whose radius is δ or denoted as $N((a,b),\delta) = \{(x,y) : \sqrt{(x-a)^2 + (y-b)^2} < \delta\}, \delta > 0$ read as neighborhood (nbd) of (a, b).

Sometimes we can also take an open rectangle centered at (a, b)

$$\{(x,y) : |x - a| < h, |y - b| < k\}, (h > 0, k > 0)$$

This is denoted as (a - h, a + h ; b - k, b + k).

Definition : A function of 2 variables is a rule that assigns to each ordered paired of real numbers (x, y) in a set D a unique real number denoted by f(x, y). The set D is the domain of f and its range is the set of values that f takes on, i.e., the set $\{f(x, y) | (x, y) \in D\}$.

Remark : We often write z = f(x, y). $D \subset \mathbf{R}^2$ and the range is a subset of **R**.
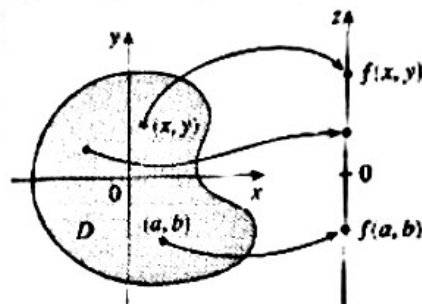
Level Curves

Definition : The level curves (or contour lines) of a function f(x, y) are the curves with equations f(x, y) = k where k is a constant in the range of f.
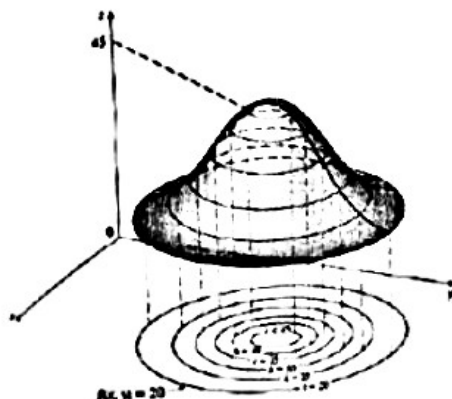
2. LIMIT OF A FUNCTION OF TWO VARIABLES

Definition. Let $f : D \to \mathbf{R}$,

The function $f(x, y)$ has a limit l ($l \in \mathbf{R}$) as (x, y) tends to $(a, b) \in \mathbf{R}^2$, written as

$$\lim_{(x, y) \to (a, b)} f(x, y) = l,$$

if given any $\varepsilon > 0$, there exists some $\delta > 0$, such that

$| f(x, y) - l | < \varepsilon$ if $0 < \sqrt{[(x - a)^2 + (y - b)^2]} < \delta$.

Equivalently,

$| f(x, y) - l | < \varepsilon$ if $0 < | x - a | < \delta, < 0 < | y - b | < \delta.$

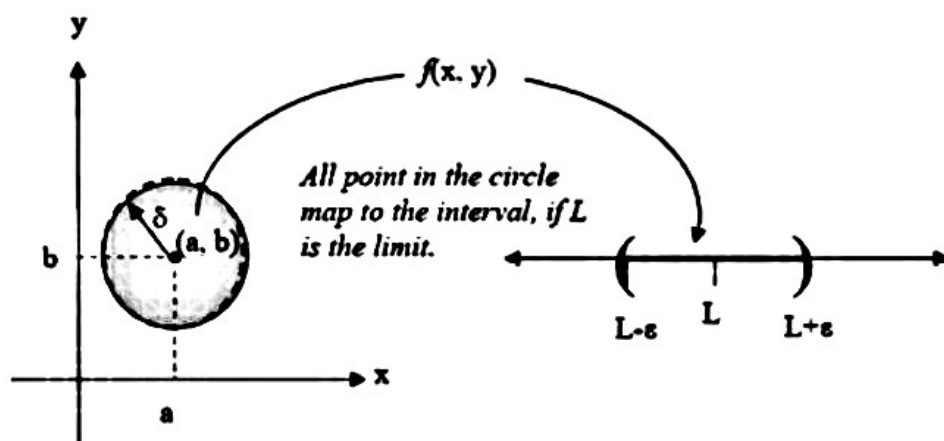*All point in the circle map to the interval, if L is the limit.*

Fig. : Illustrates This

Note : In the above definition, the function f may or may not be defined at the point (a, b).

Repeated Limits

Let $f : D \to \mathbf{R}$, D If $\lim_{y \to b} f(x, y)$ exists, then it is a function of x say $h(x)$, which is defined on some

subset of $\mathbf{R}$. If $\lim_{x \to a} h(x)$ exists, then we say that the repeated limit $\lim_{x \to a} \lim_{y \to b} f(x, y)$ exists. Similarly we can

define the repeated limit $\lim_{y \to b} \lim_{x \to a} f(x, y)$.

Remarks

(1)   Repeated limits may exist but may not be equal.

(2)   Repeated limits may exist and be equal but the two-variable limit may not exist.

(3)   Repeated limits and the two-variable limit may all exist and may be equal.

(4)   Repeated limits may not exist but the two-variable limit may exist.

Note : To find the limit of a function of one variable, we only needed to test the approach from the left as well as from the right. If both approaches were the same, the function had a limit. To find the limit of a function of two variables however, we must show that the limit is the same no matter from which direction we approach (a, b)

Example

The function f defined by

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad x^2 + y^2 \neq 0$$

$\lim\limits_{x \to 0} \lim\limits_{y \to 0} f(x, y)$ and $\lim\limits_{y \to 0} \lim\limits_{x \to 0} f(x, y)$ both exist but are unequal. Also show that $\lim\limits_{(x, y) \to (0, 0)} f(x, y)$ does not exist.

$$\lim\limits_{x \to 0} \frac{x^2 - y^2}{x^2 + y^2} = -1, \qquad \lim\limits_{y \to 0} \frac{x^2 - y^2}{x^2 + y^2} = 1.$$

$$\therefore \qquad \lim\limits_{y \to 0}\lim\limits_{x \to 0} \frac{x^2 - y^2}{x^2 + y^2} = -1, \quad \lim\limits_{x \to 0}\lim\limits_{y \to 0} \frac{x^2 - y^2}{x^2 + y^2} = 1.$$

Thus the two repeated limits exist but are unequal.

Suppose $(x, y) \to (0, 0)$ along the path $y = mx$.

Now $\quad f(x, mx) = \dfrac{x^2 - m^2 x^2}{x^2 + m^2 x^2}$

$\Rightarrow \qquad \lim\limits_{x \to 0} f(x, mx) = \dfrac{1 - m^2}{1 + m^2},$

which is different for different values of m.

Hence $\lim\limits_{(x, y) \to (0, 0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Ex. Find the limit.

$$\lim\limits_{(x,y) \to (1,2)} \frac{5x^2 y}{x^2 + y^2}$$

Notice that the point (1, 2) does not cause division by zero or other domain issues. So,

$$\lim\limits_{(x,y) \to (1,2)} \frac{5x^2 y}{x^2 + y^2} = \frac{5(1)^2(2)}{(1)^2 + (2)^2} = \frac{10}{5} = 2.$$

Ex. Find the limit.

$$\lim\limits_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2}$$

approaching (0,0) along x=0 i.e. y- axis

as x = 0 $\qquad \dfrac{x^2}{x^2 + y^2} = \dfrac{0}{0 + y^2} \to 0$

Similarly, approaching (0,0) along y=0 i.e. x- axis

as y = 0: $\qquad \dfrac{x^2}{x^2 + 0} = \dfrac{x^2}{x^2} = 1$

Since we got two different results, the limit does not exist.

Ex. Find the limit.

$$\lim\limits_{(x,y) \to (0,0)} \frac{x^2 - 2y^2}{x^2 + y^2}$$

Let x = 0: then $\lim\limits_{y \to 0} \dfrac{-2y^2}{y^2} = -2$

Now let y = 0; then $\lim\limits_{x \to 0} \dfrac{x^2}{x^2} = 1$

Again, the limit does not exist.

Ex. Find the limit.

$$\lim\limits_{(x,y) \to (0,0)} \dfrac{xy}{x^2 + y^2}$$

Let (x,y) → (0,0) along the line y = mx. Then $\dfrac{xy}{x^2 + y^2} = \dfrac{mx^2}{x^2 + m^2 x^2} = \dfrac{m}{1 + m^2}$. This shows that the

limit depends on the choice of m. Therefore, the limit does not exist.

NOTE : When we use the definition of limit to show that a particular limit exists, we usually employ certain key or basic inequalities such as:

$$|x| < \sqrt{x^2 + y^2} \qquad\qquad |y| < \sqrt{x^2 + y^2}$$

$$\dfrac{x}{x+1} < 1 \qquad\qquad \dfrac{x^2}{x^2 + y^2} < 1$$

$$|x - a| = \sqrt{(x - a)^2} \le \sqrt{(x - a)^2 + (y - a)^2}$$

and the algebraic terms.

Ex. Using ε − δ definition of limit, prove that:

$$\lim\limits_{(x,y) \to (1,2)} (3x + 2y) = 7$$

Sol. Given ε > 0, we shall show that there exist δ > 0 such that

$$0 < \sqrt{\left[(x-1)^2 + (y-2)^2\right]} < \delta \Rightarrow |3x + 2y - 7| < \varepsilon$$

Now by the triangle inequality,

$$|3x + 2y - 7| = |5x - 3 + 2y - 4| \le |3x - 3| + |2y - 4| = 3|x - 1| + 2|y - 2|$$

Since $\quad |x - 1| = \sqrt{\left[(x - 1)^2\right]} \le \sqrt{\left[(x - 1)^2 + (y - 2)^2\right]}$

and $\quad |y - 2| = \sqrt{\left[(y - 2)^2\right]} \le \sqrt{\left[(x - 1)^2 + (y - 2)^2\right]}$

Therefore $\quad |3x + 2y - 7| \le 3|x - 1| + 2|y - 2|$

$$\le 5\sqrt{\left[(x - 1)^2 + (y - 2)^2\right]}$$

Now let $\quad \delta = \dfrac{\varepsilon}{5}$; then if $\sqrt{\left[(x - 1)^2 + (y - 2)^2\right]} < \delta$

$$|3x + 2y - 7| \le 3|x - 1| + 2|y - 2|$$

$$\le 5\sqrt{\left[(x - 1)^2 + (y - 2)^2\right]}$$

$$< 5\delta = 5.\frac{\varepsilon}{5} = \varepsilon$$

Therefore δ exists.

Hence $\displaystyle\lim_{(x,y)\to(1,2)} (3x + 2y) = 7$.

Ex. Using $\varepsilon - \delta$ definition of limit, prove that:

$$\lim_{(x,y)\to(5,-2)} (ax + by) = 5a - 2b$$

Sol. Given $\varepsilon > 0$, we shall show that there exist $\delta > 0$ such that

$$0 < \sqrt{[(x - 5)^2 + (y + 2)^2]} < \delta \Rightarrow |(ax + by) - (5a - 2b)| < \varepsilon$$

Now by the triangle inequality

$$|(ax + by) - (5a - 2b)| = |a(x - 5) + b(y + 2) \leq a|x - 5| + b|y + 2|$$

Clearly, $\qquad \sqrt{[(x - 5)^2 + (y + 2)^2]} < \delta \Rightarrow |x - 5| < \delta, |y + 2| < \delta$

$\therefore \qquad |(ax + by) - (5a - 2b) \leq a|x - 5| + b|y + 2| < a\delta + b\delta = (a + b)\delta$

Now choose $\delta = \dfrac{\varepsilon}{a + b}$

$$|(ax + by) - (5a - 2b)| < (a + b)\frac{\varepsilon}{(a + b)} < \varepsilon$$

Therefore δ exists.

Hence $\displaystyle\lim_{(x,y)\to(5,-2)} (ax + by) = 5a - 2b$

Ex. Using $\varepsilon - \delta$ technique, prove that:

$$\lim_{(x,y)\to(2,3)} xy = 6$$

Sol. Given $\varepsilon > 0$, we shall show that there exist $\delta > 0$ such that

$$0 < \sqrt{[(x - 2)^2 + (y - 3)^2]} < \delta \Rightarrow |xy - 6| < \varepsilon$$

Now by the triangle inequality,

$$|xy - 6| = |xy - 3x + 3x - 6| = |x(y - 3) + 5(x - 2)| \leq |x| |y - 3| + 3|x - 2|$$

Clearly, $\qquad \sqrt{[(x - 2)^2 + (y - 3)^2]} < \delta \Rightarrow |x - 2| < \delta, |y - 3| < \delta$

Now choose $\delta = 1$,

$\qquad |x - 2| < 1 \Rightarrow 1 < x < 3 \Rightarrow 1 < |x| < 3$

Therefore $\qquad |xy - 6| \leq |x||y - 3| + 3|x - 2|$

$\qquad\qquad\qquad < 3|y - 3| + 3|x - 2|$

$\qquad\qquad\qquad < 3\delta + 3\delta = 6\delta$

Therefore, for $\delta = (\varepsilon/6)$, $|xy - 6| < 6. (\varepsilon/6) = \varepsilon$

Therefore δ exists.

Hence $\displaystyle\lim_{(x,y)\to(2,3)} xy = 6$

Ex. Using $\varepsilon - \delta$ definition of limit, prove that:

$$\lim_{(x,y)\to(1,1)}(x^2 + 2y) = 3$$

Sol. Given $\varepsilon > 0$, we shall show that there exits $\delta > 0$ such that

$$0 < \sqrt{[(x-1)^2 + (y-1)^2]} < \delta \Rightarrow |x^2 + 2y - 3| < \varepsilon$$

Now by the triangle inequality,

$$|x^2 + 2y - 3| = |x^2 - 1 + 2y - 2| = |x^2 - 1 + 2(y-1)|$$

$$\leq |x^2 - 1| + 2|y - 1|$$

$$\leq |x - 1||x + 1| + 2|y - 1|$$

Clearly, $\quad \sqrt{[(x-1)^2 + (y-1)^2]} < \delta \Rightarrow |x-1| < \delta, |y-1| < \delta$

Now choose $\delta = 1$

$$|x - 1| < 1 \Rightarrow 0 < x < 2$$
$$\Rightarrow 1 < x + 1 < 3 \Rightarrow 1 < |x + 1| < 3$$

$\therefore \quad |x^2 + 2y - 3| \leq |x - 1| \, |x + 1| + 2|y - 1|$
$$< 3|x - 1| + 2|y - 1|$$
$$< 3\delta + 2\delta = 5\delta$$

Now choose $\delta = (\varepsilon / 5)$, $|x^2 + 2y - 3| < (5\varepsilon / 5) < \varepsilon$

Therefore required $\delta = \min\{1, (\varepsilon/5)\}$ i.e., δ exists.

Hence $\lim_{(x,y)\to(1,1)}(x^2 + 2y) = 3$

3. CONTINUITY OF A FUNCTION OF TWO VARIABLES

Definition : Let $f : D \rightarrow R$

Then f is said to be continuous at a point $(a, b) \in D$ if

$$\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$$

In other words, f is continuous at a point $(a, b) \in D$ if given and $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$| f(x, y) - f(a, b) | < \varepsilon, \text{ when } \sqrt{\left[(x - a)^2 + (y - b)^2 \right]} < \delta.$$

OR

$$| f(x, y) - f(a, b) | < \varepsilon, \text{ when } | x - a | < \delta, | y - b | < \delta.$$

Ex. Examine the following function for continuity at (0,0):

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Sol. We know that if $\lim_{(x,y) \to (0,0)} f(x, y)$, $(a, b) \in R^2$ exists, then this limit is independent of path of approach along which we approach the point (a, b).

Since $f(x, y)$ is identically zero along the co-ordinates,

therefore when $(x, y) \to (0, 0)$, then the limit along each axis is zero.

But the path of approach is along a straight line $y = mx$, where $m \neq 0$,

$$\text{then} \quad f(x, y) = \frac{mx^2}{x^2 + m^2 x^2} = \frac{m}{1 + m^2} (x \neq 0)$$

$$\therefore \quad \lim_{(x,y) \to (0,0)} f(x, y) \lim_{(x,y) \to (0,0)} f(x, y) = \frac{m}{1 + m^2}, \text{ which depends on m., i.e., the limit of the function}$$

depends on the path of approach.

Therefore $\lim_{(x,y) \to (0,0)} f(x, y)$ does not exist.

Again if the path of approach be parabola $x = y^2$, then

$$\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{y \to 0} \frac{y^3}{y^4 + y^2} = \lim_{y \to 0} \frac{y}{y^2 + 1} = 0 \qquad \dots(2)$$

Since the limits obtained by two different approaches are different, so

$$\lim_{(x,y) \to (0,0)} f(x, y) \text{ does not exist.}$$

Hence the function $f(x, y)$ is not continuous at the point (0,0).

Ex. Examine for continuity the following function at (0,0) :

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^2}, & x \neq 0, y \neq 0 \\ 1, & x = 0, y = 0 \end{cases}$$

Sol. Given $f(0,0) = 1$ ...(1)

and $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$ ...(2)

Since $(1) \neq (2)$

Therefore the given function is discontinuous at $(0,0)$.

Ex. Show that the following function $f : \mathbf{R}^2 \to \mathbf{R}$ is continuous at $(0,0)$:

$$f(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Sol. Let $\varepsilon > 0$ and $|x - 0| < \sqrt{\varepsilon}$ and $|y - 0| < \sqrt{\varepsilon}$, then

$$|f(x,y) - f(0,0)| = \left| \frac{xy(x^2 - y^2)}{x^2 + y^2} - 0 \right| = |xy| \left| \frac{x^2 - y^2}{x^2 + y^2} \right|$$

$$\leq |xy| \left\{ \because |x^2 - y^2| \leq |x^2 + y^2|, \therefore \left| \frac{x^2 - y^2}{x^2 + y^2} \right| \leq 1 \right\}$$

$$\leq |x| \, |y|$$

$$< \sqrt{\varepsilon}\sqrt{\varepsilon} = \varepsilon$$

i.e., $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists and equal to 0.

And value of the $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$.

Hence the given function is continuous at $(0, 0)$.

Ex. Show that the real valued function $f(x,y)$ of two variables defined below is continuous at the origin:

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Sol. Let $x = r\cos\theta$, $y = r\sin\theta$ and $\varepsilon > 0$,

then $|f(x,y) - f(0,0)| = \left| \dfrac{xy}{\sqrt{x^2 + y^2}} \right| = \left| \dfrac{r^2\cos\theta\sin\theta}{r} \right|$

$$= r|\cos\theta\sin\theta| \leq r \; [\because |\cos\theta\sin\theta| \leq 1]$$

Now $|f(x,y) - f(0,0)| < \varepsilon$ if $r < \varepsilon$ or, if $\sqrt{x^2 + y^2} < \varepsilon$

Here if we take $\delta < \dfrac{\varepsilon}{\sqrt{2}}$, then $|x| < \delta$ and $|y| < \delta$

$$\Rightarrow |x| < \frac{\varepsilon}{\sqrt{2}} \text{ and } |y| < \frac{\varepsilon}{\sqrt{2}} \Rightarrow x^2 + y^2 < \varepsilon^2 \Rightarrow \sqrt{x^2 + y^2} < \varepsilon$$

Thus, for every $\varepsilon > 0$, $\exists \delta = \dfrac{\varepsilon}{\sqrt{2}} > 0$ such that

$|f(x, y) - f(0, 0)| < \varepsilon$ whenever $|x - 0| < \delta$ and $|y - 0| < \delta$

Therefore f is continuous at the origin (0,0).

Theorem : The following results are true for multivariable functions:

1. The sum, difference and product of continuous functions is a continuous function.

2. The quotient of two continuous functions is continuous as long as the denominator is not 0.

3. Polynomial functions are continuous.

4. Rational functions are continuous in their domain.

5. If f(x,y) is continuous and g(x) is defined and continuous on the range of f, then g(f(x,y)) is also continuous.

Ex. Show that

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous at (0,0).

Sol. This function is clearly continuous everywhere except at (possibly) (0,0).

Let's check continuity at (0,0).

Let $\varepsilon > 0$ be given. Let $\delta = \dfrac{\varepsilon}{2}$ and suppose that $0 < \sqrt{x^2 + y^2} < \delta$.

Then, using the fact that $x^2 \leq x^2 + y^2$, i.e. $\left| \dfrac{x^2}{x^2 + y^2} \right| \leq 1$ we have

$$|f(x,y) - 0| = \left| \dfrac{x^2 y}{x^2 + y^2} \right| = \left| \dfrac{x^2}{x^2 + y^2} \right| |y| \leq |y| \leq \sqrt{x^2 + y^2} < \delta = \dfrac{\varepsilon}{2} < \varepsilon.$$

Hence, f(x,y) is continuous at (0,0).

Ex. Show that

$$f(x,y) = \begin{cases} \dfrac{x^2}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is discontinuous at (0,0).

Sol. Again this function is clearly continuous everywhere except (possibly) at (0,0). Now let's look at the limit as (x, y) approaches (0, 0) along two different paths. First, let's approach (0,0) along the x-axis, i.e. y = 0.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,0)\to(0,0)} \dfrac{x^2}{x^2 + 0} = 1.$$

Now, let's approach (0,0) along the y-axis, i.e. x = 0.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(0,y)\to(0,0)} \dfrac{0}{0 + y^2} = 0.$$

Since the limit is not the same along the two different directions we conclude that $f(x, y)$ is discontinuous at $(0, 0)$.

Ex. The function $f(x, y) = \dfrac{5x^2 y}{x^2 + y^2}$ is not continuous at $(0, 0)$. However, the limit at $(0, 0)$ does exist and is equal to zero at the that point. If we define $f(x, y)$ to be equal to zero at $(0, 0)$, then we have removed the point of discontinuity and the function becomes continuous at that point.

If $f(x, y)$ and $g(x, y)$ are two functions that are continuous at (x_0, y_0), then the function $\dfrac{f(x,y)}{g(x,y)}$ is also continuous at (x_0, y_0) only if $g(x_0, y_0) \neq 0$.

Ex. $f(x, y) = 5x^2 y$ is continuous at $(0, 0)$

$g(x, y) = x^2 + y^2$ is continuous at $(0, 0)$

However, $\dfrac{f(x,y)}{g(x,y)}$ is not continuous at $(0, 0)$ because $g(0, 0) = 0$.

Ex. Find where $\tan^{-1}\left(\dfrac{xy^2}{x+y}\right)$ is continuous.

Here, we have the composition of two functions. We know that $\tan^{-1} x$ is continuous on its domain, that is on $\square$. Therefore, $\tan^{-1}\left(\dfrac{xy^2}{x+y}\right)$ will be continuous where $\dfrac{xy^2}{x+y}$ is continuous. Since $\dfrac{xy^2}{x+y}$ is the quotient of two polynomial functions, therefore it will be continuous as long as its denominator is not 0, that is as long as $y \neq -x$. It follows that $\tan^{-1}\left(\dfrac{xy^2}{x+y}\right)$ is continuous on $\{(x, y) \in \square^2 \mid y \neq -x\}$.

Ex. Find where $\ln(x^2 + y^2 - 1)$ is continuous.

Again, we have the composition of two functions. It is continuous where it is defined, that is on $\{x \in \mathbf{R} \mid x > 0\}$. So, $\ln(x^2 + y^2 - 1)$ will be continuous as long as $x^2 + y^2 - 1$ is continuous and positive. $x^2 + y^2 - 1$ is continuous on $\mathbf{R}^2$, but $x^2 + y^2 - 1 > 0$ if and only if $x^2 + y^2 > 1$, that is outside the circle of radius 1, centered at the origin. It follows that $\ln(x^2 + y^2 - 1)$ is continuous of the portion of $\mathbf{R}^2$ outside the circle of radius 1, centered at the origin.

4. PARTIAL DERIVATIVES

If $f(x) = 5x^3$, then $f'(x) = 5 \times 3x^2 = 15x^2$. In the same way, if you're given $g(x) = ax^3$, and told that a is a constant, then you find that $g'(x) = a \cdot 3x^2 = 3ax^2$. If you are now told that $a = 5$, you can plug in 5 for a in this latter answer to get what you got before.

Suppose now that you're given the function of two variables $h(x,y) = yx^3$. Since y is one of the independent variables in h, clearly y is not intended to always be constant. However. if you're told to assume that, for some physical or mathematical reason, y is held constant at the value $y = 5$, and asked to differentiate h as a function of x, you would look at $h(x, 5) = 5x^3$, and differentiate, to once again obtain $15x^3$. If, instead, you're told to assume that y is held constant at the value $y = 7$, and asked to differentiate h as a function of x, you would look at $h(x, 7) = 7x^3$, and differentiate to obtain $21x^3$.

More generally, you could just be told to assume that y is held constant, without being told that the constant value is 5 or 7 or anything else specific; then, you can calculate the derivative of yx^3, with respect to x, thinking of y as a constant; you find $y \cdot 3x^2$.

This process of taking the derivative, with respect to a single variable, and holding constant all of the other independent variables, is called finding (or, taking) a partial derivative. This is a fundamental mathematical concept that arises in many contexts.

Definition : Suppose that we have a real-valued function $z = f(x,y)$ of two real variables. Then, the derivative of f, with respect to x, holding y constant, is called the partial derivative of f, with respect to x, and is denoted by any of

$$\frac{\partial z}{\partial x}, \qquad \frac{\partial f}{\partial x}, \qquad f_x(x,y), \qquad \text{or} \qquad f_1(x,y).$$

In the same way, the derivative of f, with respect to y. Holding x constant, is called the partial derivative of f, with respect to y, and is denoted by any of

$$\frac{\partial z}{\partial y}, \qquad \frac{\partial f}{\partial y}, \qquad f_y(x,y), \qquad \text{or} \qquad f_2(x,y).$$

We also use the partial derivative operators:

$$\frac{\partial}{\partial x} \qquad \text{and} \qquad \frac{\partial}{\partial y},$$

which tell you to take the partial derivative with respect to x and y, respectively.

We mention the notations $f_1(x,y)$ and $f_2(x,y)$ primarily because you may see them used in other books; there are also technical reasons why these notations are useful in some contexts. However, we shall avoid their use to indicate partial derivatives, since we like to reserve the notations $f_1(x,y)$ and $f_2(x,y)$ for use in denoting the component functions of a multi-component function $f(x,y) = (f_1(x,y), f_2(x,y))$. In any case, we will explicitly state, or the context will make clear, what we mean by f_1, f_2 or, more generally, f_i.

Example. Consider the fairly simple function

$$z = f(x,y) = x^2 - y^2.$$

We first take the partial derivative, with respect to x, thinking of y as a constant. Let's use all of our various notations just for practice. We find

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x(x,y) = f_1(x,y) = \frac{\partial}{\partial x}(x^2 - y^2) = 2x.$$

Now, we take the partial derivative, with respect to y, thinking of x as a constant.

We find

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y(x,y) = f_2(x,y) = \frac{\partial}{\partial y}(x^2 - y^2) = -2y.$$

Example. Of course, if you have a function, such as h(t) = 5t + ln t, which depends on only one variable, the partial derivative is just the same as the ordinary derivative:

$$\frac{\partial h}{\partial t} = \frac{dh}{dt} = 5 + \frac{1}{t}.$$

It's not wrong to write the partial derivative here, but it could be misleading in some cases; it might make someone wonder what the other variables are.

Example. Find the partial derivatives of $xy^2 + 5y^3$.

First, calculate the partial derivative with respect to x, by thinking of y as constant:
we find

$$\frac{\partial}{\partial x}(xy^2 + 5y^3) = 1 \cdot y^2 + 0 = y^2.$$

Now, calculate the partial derivative with respect to y, by thinking of x as constant;
we find

$$\frac{\partial}{\partial y}(xy^2 + 5y^3) = x \cdot 2y + 15y^2 = 2xy + 15y^2.$$

(Another Definition) Let $f : D \rightarrow \mathbf{R}$, and (a, b) be any point of D.

The partial derivative of f w.r.t. x at the point (a, b), denoted as $f_x(a, b)$, is defined as

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h},$$

provided the limit exists.

The partial derivative of f w.r.t. y at the point (a, b), denoted as $f_y(a, b)$, is defined as

$$f_y(a,b) = \lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k},$$

provided the limit exists.

$f_x(a, b)$, $f_y(a, b)$ are called the first order partial derivatives of f at (a, b).

Now the second order partial derivatives of f at (a, b) viz.

$f_{xx}(a, b)$, $f_{yx}(a, b)$, $f_{xy}(a, b)$, $f_{yy}(a, b)$.

Defined as

$$f_{xx}(a, b) = \lim_{h \to 0} \frac{f_x(a+h, b) - f_x(a, b)}{h}, \text{ provided the limit exists.}$$

$$f_{yx}(a, b) = \lim_{k \to 0} \frac{f_x(a, b+k) - f_x(a, b)}{k}, \text{ provided the limit exists.}$$

$$f_{xy}(a, b) = \lim_{h \to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h}, \text{ provided the limit exists,}$$

$$f_{yy}(a, b) = \lim_{k \to 0} \frac{f_y(a, b+k) - f_y(a, b)}{k}, \text{ provided the limit exists.}$$

If function is continuous then $f_{xy}(a, b) = f_{yx}(a, b)$.

In general, $f_{xy}(a,b) \neq f_{yx}(a,b)$

Ex. The volume, V, of a right circular cylinder is given by

$$V = \pi r^2 h,$$

where r is the radius of the base, and h is the height. Suppose that the cylinder is some sort of container for which the height can vary, such as the interior of a piston.

What is the instantaneous rate of change of the volume, with respect to the height, when the height is 0.3 meters, if the radius is held constant at 0.1 meters?

Sol. We hold r constant and find

$$\frac{\partial V}{\partial h} = \pi r^2,$$

in cubic meters per meter (or, square meters).

Thus, the instantaneous rate of change of the volume, with respect to the height, when the height is 0.3 m, and the radius is held constant at 0.1 m is

$$\frac{\partial V}{\partial h}\bigg|_{(r,h)=(0.1,\,0.3)} = \pi(0.1)^2 = 0.01\pi \; m^3/m.$$

Note that this result is independent of h, so that, in the end, we don't need to use the data that h = 0.3 meters.

Example. Suppose that we have the function

$$w = f(x,y,z) = x\sin(yz) + y^2 e^z + x^3.$$

Then, we find:

$$\frac{\partial w}{\partial x} = \sin(yz) + 0 + 3x^2,$$

$$\frac{\partial w}{\partial y} = x(\cos(yz))z + 2ye^z + 0,$$

and

$$\frac{\partial w}{\partial z} = x(\cos(yz))y + y^2 e^z + 0.$$

Theorem : If U is a non-empty connected open subset of R^n, and f is a function on U such that all of the partial derivatives of f exist and are 0 at each point in U, then f is constant on U.

Definition : The multi-component function

$$\vec{\nabla}f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

of partial derivatives of a function $f = f(x_1,\ldots,x_n)$ is called the **gradient vector (function)** of f.

Its value at a point **p** is denoted either by $\vec{\nabla}f(p)$ or by $\vec{\nabla}f_{|p}$.

Example. Consider $f(x,y) = x^2 - y^2$. Then,

$$\vec{\nabla}f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x, -2y) = 2(x, -y),$$

and

$$\vec{\nabla}f(3,4) = (6,-8) = 2(3,-4).$$

Higher-order partial derivatives :

Consider the function

$$f(x,y) = x^2 + 5xy - 4y^2.$$

The partial derivatives are easy to calculate:

$$\frac{\partial f}{\partial x} = 2x + 5y \quad \text{and} \quad \frac{\partial f}{\partial y} = 5x - 8y.$$

We now want to look at the second partial derivatives of f.

A second partial derivative should be a partial derivative of a partial derivative. So, there are four second partial derivatives: you can first take two different first partial derivatives, with respect to x or y and then, for each of those, you can take a partial derivative a second time with respect to x or y.

We introduce new notation, and calculate

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x + 5y) = 2,$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x + 5y) = 5,$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(5x - 8y) = 5,$$

and $\quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(5x - 8y) = -8.$

The two second partial derivatives f_{xy} and f_{yx} above, the ones with one partial derivative with respect to x and one with respect to y, are called mixed partial derivatives.

Note that f_{xy} and f_{yx} are equal in this example. While this is not **always** the case, it's true for "most" of the functions that we deal with. More precisely, we need the continuity condition given in the following theorem.

Theorem : Suppose that f, f_x, f_y, and f_{xy} exist in an open ball around a point (x_0, y_0), and that f_{xy} is continuous at (x_0, y_0). Then, $f_{yx}(x_0, y_0)$ exists and

$$f_{yx}(x_0, y_0) = f_{xy}(x_0, y_0).$$

Example. Let's try calculating some more-complicated partial derivatives.

Suppose that

$$z = f(x,y) = x \sin(xy) + 3y^4.$$

We want to calculate the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$. Note that x sin(xy) is the product of two functions of x, which will require the Product Rule when differentiating with respect to x. However, when we take the partial derivative with respect to y, we should think of x sin(xy) as being a "constant" times a function of y; hence, we will not need the Product Rule when applying $\partial/\partial y$.

We find :

$$\frac{\partial f}{\partial x} = x \cdot \frac{\partial}{\partial x}(\sin(xy)) + \sin(xy) \cdot \frac{\partial}{\partial x}(x) + 0 = x\cos(xy) \cdot \frac{\partial}{\partial x}(xy) + \sin(xy)$$

$$= [x\cos(xy)] \cdot y + \sin(xy) = xy\cos(xy) + \sin(xy),$$

and

$$\frac{\partial f}{\partial y} = x \cdot \frac{\partial}{\partial y}(\sin(xy)) + 12y^3 = x\cos(xy) \cdot \frac{\partial}{\partial y}(xy) + 12y^3$$

$$= x\cos(xy) \cdot x + 12y^3 = x^2\cos(xy) + 12y^3.$$

Ex. Suppose that $g(x,y) = xe^y + y^2 \tan^{-1}x$. Calculate the dot product.

$$\vec{\nabla}g(1,1) \cdot (2,-2).$$

Sol. We find $\vec{\nabla}g = \left(e^y + \dfrac{y^2}{1+x^2}, xe^y + 2y\tan^{-1}x \right),$

and so $\vec{\nabla}g(1,1) = \left(e + \dfrac{1}{2}, e + 2 \cdot \dfrac{\pi}{4} \right) = \left(e + \dfrac{1}{2}, e + \dfrac{\pi}{2} \right).$

Therefore,

$$\vec{\nabla}g(1,1) \cdot (2,-2) = \left(e + \frac{1}{2}, e + \frac{\pi}{2} \right) \cdot (2,-2)$$

$$= 2\left(e + \frac{1}{2} \right) + (-2)\left(e + \frac{\pi}{2} \right) = 1 - \pi.$$

Definition : Suppose that $f(x_1, x_2,...,x_n)$ is a real-valued function whose domain is a subset of $\mathbf{R}^n$. Then, we define the partial derivative of f with respect to x_i to be

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1,...,x_{i-1}, x_i + h, x_{i+1},...x_n) - f(x_1,...,x_{i-1}, x_i, x_{i+1},...,x_n)}{h},$$

provided that this limit exists. If the Limit fails to exist, then we say that the partial derivative is undefined.

Recalling that e_i denotes the i-th standard basis element and letting $x = (x_1, x_2,...,x_n)$, then the definition above is equivalent to

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x + he_i) - f(x)}{h}.$$

Theorem : If u is a non-empty connected open subset of $\mathbf{R}^n$, and f and g are two functions on u such that all of the partial derivative of f and g exist and are equal at each point in u, then f and g differ by a constant on u, i.e., there exists a constant C such that, for all p in u, f(p) = g(p) + C.

Theorem : Suppose that $f(x_1, x_2,...,x_n)$ is a real-valued function such that f and all of its partial derivatives of order less than or equal to r are defined and continuous on an open subset u of $\mathbf{R}^n$. Then, at each point p in u, every partial derivative of order r is independent of the order in which the partial derivatives are calculated.

Example.   Suppose that

$f(x,y,z) = x^5y^6z^7 + xe^{yz} + y^2 \sin(3x - 5z).$

Then, Theorem implies that

$$\frac{\partial^4 f}{\partial z \partial y \partial x^2} = \frac{\partial^4 f}{\partial z \partial x \partial y \partial x} = \frac{\partial^4 f}{\partial x^2 \partial y \partial z},$$

which are equal to every other 4th order partial derivative that's with respect to x twice, and y and z once each.

They're all equal to

$$840x^3y^5z^6 + 90y\cos(3x - 5x).$$

Example. Let

$$f(r, \theta) = (r\cos\theta, r\sin\theta).$$

Then

$$f_r = (\cos\theta, \sin\theta),$$
$$f_\theta = (-r\sin\theta, r\cos\theta),$$
$$f_{rr} = (0, 0)$$
$$f_{\theta\theta} = (-r\cos\theta, -r\sin\theta),$$

and

$$f_{r\theta} = f_{\theta r} = (-\sin\theta, \cos\theta).$$

5. DIFFERENTIABILITY OF TWO VARIABLES

Definition : Let $f : D \to R$ be a real-valued function defined. Then f is said to be differentiable at a point $(a, b) \in D$, if

$$f(a + h, b + k) - f(a, b) = Ah + Bk + \sqrt{h^2 + k^2}\ g(h, k),$$

where A and B are real numbers independent of h and k, and g is a real-valued function such that $\lim_{(h, k) \to (0, 0)} g(h, k) = 0.$

Equivalently, f is differentiable at (a, b), if

$$f(a + h, b + k) - f(a, b) = Ah + Bk + h\phi(h, k) + k\psi(h, k)$$

where A and B are real numbers independent of h and k and $\phi(h, k)$, $\psi(h, k) \to 0$ as $(h, k) \to (0, 0)$.

Theorem : If a function $f : D \to R$ is differentiable at $(a, b) \in D$, then f has partial derivatives f_x and f_y at (a, b).

Proof. Since f is differentiable at the point $(a, b) \in D$, so there exist real numbers A and B, and a function g such that

(i) $\qquad f(a + h, b + k) - f(a, b) = Ah + Bk + \sqrt{h^2 + k^2}\ g(h, k),$ ...(1)

(ii) $\qquad$ A and B are independent of h and k, and

(iii) $\qquad \lim_{(h, k) \to (0, 0)} g(h, k) = 0.$

Putting k = 0 in (1), we have

$$f(a + h, b) - f(a, b) = Ah + |h|\ g(h, 0).$$

Dividing throughout by h, and taking limits as $h \to 0$, we have

$$\lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h} = \lim_{h \to 0}\left[A + \frac{|h|}{h} g(h, 0)\right]$$

$$= A + (\pm 1).0 = A, \text{ since } g(h, 0) \to 0 \text{ as } h \to 0.$$

$\therefore \qquad f_x(a, b) = A.$

Putting h = 0 in (1), we have

$$f(a, b + k) - f(a, b) = Bk + |k|\ g(0, k).$$

Dividing throughout by k, and taking limits as $k \to 0$, we have

$$\lim_{k \to 0} \frac{f(a, b + k) - f(a, b)}{k} = B.$$

$\therefore \qquad f_y(a, b) = B.$

Hence f_x and f_y exist at the point (a, b).

Remark

1. It may be noted that the constants A, B appearing in the definition of differentiability of f are given by :
$$A = f_x(a, b),\ B = f_y(a, b).$$

2. If either of f_x, f_y does not exist at (a, b), then f is not differentiable at (a, b).

Ex. Let $f : R^2 \to R$ be defined by setting.

$f(x, y) = xy / \sqrt{(x^2 + y^2)}$, when $(x, y) \neq (0, 0)$, $f(0, 0) = 0$.

f_x and f_y exist at (0, 0) but f is not differentiable at (0, 0). Also f is continuous at (0, 0).

Sol. $f_x(0, 0) = \lim\limits_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{0 - 0}{h} = 0$

$f_y(0, 0) = \lim\limits_{k \to 0} \dfrac{f(0, k) - f(0, 0)}{k} = \lim\limits_{k \to 0} \dfrac{0 - 0}{k} = 0.$

Thus $f_x(0, 0)$ and $f_y(0, 0)$ both exist and are equal to 0.

Let, if possible, f be differentiable at (0, 0). Then

$$f(h, k) - f(0, 0) = Ah + Bk + \sqrt{(h^2 + k^2)}\, g(h, k), \qquad \qquad ...(1)$$

where $\lim\limits_{(h, k) \to (0, 0)} g(h, k) \to 0,$ $\qquad \qquad ...(2)$

and $A = f_x(0, 0) = 0$, $B = f_y(0, 0) = 0$.

From (1), we obtain

$$hk/\sqrt{(h^2 + k^2)} - 0 = 0.h + 0.k + \sqrt{(h^2 + k^2)}\, g(h, k),$$

or $\qquad g(h, k) = \dfrac{hk}{h^2 + k^2}.$

Taking $k = mh$, we see that

$$\lim\limits_{h \to 0} g(h, mh) = \dfrac{m}{1 + m^2}, \text{ which depends on } m.$$

Thus $\lim\limits_{(h, k) \to (0,0)} g(h, k)$ does not exist, which contradicts (2).

Hence f is not differentiable at (0, 0) f is continuous at (0, 0).

Theorem : If f(x, y) is differentiable at a point (a, b), then it is continuous at (a, b), but converse need not be true.

i.e. Differentiability implies continuity but continuity need not imply differentiability.

Proof. Since f(x, y) is differentiable at the point (a, b), therefore

$$\lim\limits_{(h,k) \to (0,0)} \dfrac{f(a + h, b + k) - f(a,b) - h f_x(a,b) - k f_y(a,b)}{\sqrt{\left(h^2 + k^2\right)}} = 0$$

$$\Rightarrow \quad \lim\limits_{(h,k) \to (0,0)} \left[f(a + h, b + k) - f(a,b) - h f_x(a,b) - k f_y(a,b) \right] = 0$$

$$\Rightarrow \quad \lim\limits_{(h,k) \to (0,0)} f(a + h, b + k) = f(a,b)$$

which shows that f is continuous at (a, b).

Remark: If a function $f : D \to R$ is differentiable at a point $(a, b)\ \varepsilon\ D$ then all of its partial derivative exists but not conversely i.e. the existence of partial derivatives of the function f at a point does not imply it is differentiable.

Theorem : If a function $f : D \to R$ has continuous partial derivatives f_x and f_y at $(a, b) \in D$, then f is differentiable at (a, b).

Proof. Since f_x and f_y are continuous at (a, b), so f_x and f_y both exist in a nbd. $N \equiv N((a, b), \delta)$ of (a, b). Let (a + h, b + k) be any point of N. We have

$$f(a + h, b + k) - f(a, b) = [f(a + h, b + k) - f(a + h, b)] + [f(a + h, b) - f(a, b)] \quad ...(1)$$

We define two functions G and H as follows :

$$G(y) = f(a + h, y) \; \forall \; y \in [b, b + k],$$

and $\qquad\qquad H(x) = f(x, b) \qquad \forall \; x \in [a, a + h].$

Since f_y exists in N, so the function G satisfies both the conditions of Lagrange's mean value theorem in [b, b + k]. Thus there exists a $\theta_1 (0 < \theta_1 < 1)$ such that

$$G(b + k) - G(b) = kG'(b + \theta_1 k)$$

or $\qquad\qquad f(a + h, b + k) - f(a + h, b) = k f_y(a + h, b + \theta_1 k). \qquad\qquad ...(2)$

Similarly, since H satisfies both the conditions of Lagrange's mean value theorem in [a, a + h], there exists $\theta_2 (0 < \theta_2 < 1)$ such that

$$f(a + h, b) - f(a, b) = h \, f_x(a + \theta_2 h, b). \qquad\qquad ...(3)$$

Using (2) and (3) in (1), we obtain

$$f(a + h, b + k) - f(a, b) = h f_x(a + \theta_2 h, b) + k f_y(a + h, b + \theta_1 k)$$

$$= hf_x(a, b) + kf_y(a, b) + h[f_x(a + \theta_2 h, b) - f_x(a, b)] + k[f_y(a + h, b + \theta_1 k) - f_y(a, b)]$$

$$= hf_x(a, b) + kf_y(a, b) + \sqrt{(h^2 + k^2)} . F(h, k), \qquad\qquad ...(4)$$

where $F(h, k) = h[f_x(a + \theta_2 h, b) - f_x(a, b)]/\sqrt{(h^2 + k^2)}$

$$+ k[f_y(a + h, b + \theta_1 k) - f_y(a, b)]/\sqrt{(h^2 + k^2)}. \qquad\qquad ...(5)$$

Since f_x and f_y are both continuous at (a, b), so

$$\left. \begin{array}{l} \lim_{(h,k) \to (0,0)} f_x(a + \theta_2 h, b) = f_x(a, b), \\[2mm] \lim_{(h,k) \to (0,0)} f_y(a + h, b + \theta_1 k) = f_y(a,b) \end{array} \right\} \qquad\qquad ...(6)$$

From (5) and (6), we obtain

$$\lim_{(h,k) \to (0,0)} F(h, k) = 0 \qquad\qquad ...(7)$$

since $\qquad \left| \dfrac{h}{\sqrt{(h^2 + k^2)}} \right| \le 1, \; \left| \dfrac{k}{\sqrt{(h^2 + k^2)}} \right| \le 1,$

for all value of (h, k), such that $h^2 + k^2 \ne 0$.

From (4) and (7), we get

$$f(a + h, b + k) - f(a, b) = hf_x(a, b) + kf_y(a, b) + \sqrt{(h^2 + k^2)} \, F(h, k),$$

where $F(h, k) \to 0$ as $(h, k) \to (0, 0)$.

Hence f is differentiable at (a, b).

Remark : If a function f is not differentiable at (a, b), then f_x and f_y cannot be continuous at (a, b).

Theorem : If a function f(x,y) is totally differentiable the partial derivatives f_x and f_y both exist and are finite.

Proof. Let f(x,y) be totally differentiable at the point (a,b), then there exist two constants α and β such that

$$\lim_{(h,k) \to (0,0)} \frac{f(a + h, b + k) - f(a,b) - \alpha h - bk}{\sqrt{(h^2 + k^2)}} = 0$$

$\Rightarrow \qquad \lim_{k \to 0} \dfrac{f(a,b + k) - f(a,b) - bk}{k} = 0 \qquad$ [putting h = 0)

$\Rightarrow \qquad \lim_{k \to 0} \dfrac{f(a,b + k) - f(a,b)}{k} = \beta$

$$\Rightarrow \qquad f_y(a, b) = \beta$$

Similarly, $\quad f_x(a,b) = \alpha$.

Accordingly by theorem a function $f(x,y)$ can be defined at the point (a,b), if

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k) - f(a,b) - hf_x(a,b) - kf_y(a,b)}{\sqrt{(h^2 + k^2)}} = 0.$$

Total Differential :

Theorem : If $u = f(x, y)$ be any differentiable function, then

$$du = f_x(x, y)\, dx + f_y(x, y)\, dy.$$

Proof. Let $u = f(x, y)$ be any differentiable function, then

$$\delta u = (u + \delta u) - u = f(x + h, y + k) - f(x, y)$$

$$= \frac{f(x+h, y+k) - f(x, y+k)}{h}.h + \frac{f(x, y+k) - f(x, y)}{k}.k$$

Hence $h \to 0$ and $k \to 0$ i.e., taking $\sqrt{h^2 + k^2} \to 0$,

$$du = \lim_{(h,k)\to(0,0)} \frac{f(x+h, y+k) - f(x, y+k)}{h} dx + \lim_{(h,k)\to(0,0)} \frac{f(x, y+k) - f(x, y)}{k} dy$$

$$= f_x(x, y)\, dx + f_y(x, y)\, dy, \qquad \text{[by mean value theorem]}$$

Here du is the total differential of u.

Sufficient Condition for Differentiability:

Theorem : If (a, b) be a point of the domain $D \subset \mathbf{R}^2$ of a real. valued function $f(x, y)$ such that
(i) $\quad f_x(a,b)$ exists (ii) $f_y(x,y)$ is continuous at (a,b), then $f(x,y)$ is differentiable at (a,b)

Proof. Let $\varepsilon > 0$ be given. Since $f_x(x,y)$ exists at (a, b), therefore there exists a $\delta_1 > 0$ such that $|h| < \delta_1$, then

$$\left| \frac{f(a+h,b) - f(a,b)}{h} - f_x(a,b) \right| < \frac{\varepsilon}{2}$$

Let $\qquad \dfrac{f(a+h,b) - f(a,b)}{h} - f_x(a,b) = \eta,$ \hfill ...(1)

where $|\eta| < \varepsilon/2$.

Again $f_y(x, y)$ exists and continuous at (a, b), therefore by mean value theorem,

$f(a + h, b + k) - f(a + h, b) = k\, f_y(a + h, b + \theta k)\ (0 < \theta < 1)$ \hfill ...(2)

Now consider the following :

$$\left| \frac{1}{\sqrt{h^2 + k^2}} \Big[f(a+h,b+k) - f(a,b) - hf_x(a,b) - k\, f_y(a,b) \Big] \right|$$

$$= \left| \frac{1}{\sqrt{h^2 + k^2}} \Big[f(a+h,b+k) - f(a+h,b) + f(a+h,b) - hf_x(a,h,b) - f(a,b) - hf_x(a,b) - k f_y(a,b) \Big] \right|$$

$$= \left| \frac{1}{\sqrt{h^2 + k^2}} \Big[k\, f_y(a+h,b+\theta k) + h\, f_x(a,b) + \eta h - h\, f_x(a,b) - k\, f_y(a,b) \Big] \right| \text{ [by (1) and (2)]}$$

$$= \left| \frac{k}{\sqrt{h^2 + k^2}} \{ f_y(a + h, b + \theta k) - f_y(a,b) \} + \frac{h}{\sqrt{h^2 + k^2}} . \eta \right|$$

$$\leq \left| \frac{k}{\sqrt{h^2 + k^2}} \right| \left| f_y(a + b, b + \theta k) - f_y(a,b) \right| + \left| \frac{h}{\sqrt{h^2 + k^2}} \right| |\eta| \qquad ...(3)$$

Since $f_y(x, y)$ is continuous at (a, b), therefore for $|h| < \delta_2$ and $|k| < \delta_2$ there exist $\delta_2 > 0$ such that

$$| f_y(a + h, b + k) - f_y(a + b) | < \varepsilon/2 \qquad ...(4)$$

Since $h \neq 0$, $k \neq 0$, then

$$\left| \frac{h}{\sqrt{h^2 + k^2}} \right| < 1, \left| \frac{k}{\sqrt{h^2 + k^2}} \right| < 1 \qquad ...(5)$$

Let $\delta = \min(\delta_1, \delta_2)$, then $|h| < \delta$ and $|k| < \delta$,

With the help of (4) and (5), the inequality reduces to

$$\left| \frac{1}{\sqrt{h^2 + k^2}} \Big[f(a + h, b + k) - f(a,b) - h f_x(a,b) - k f_y(a,b) \Big] \right|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad [\because |\eta| < \varepsilon/2]$$

Now taking $\varepsilon \to 0$ as $h \to 0$, $k \to 0$,

$$\lim_{(h,k) \to (0,0)} \frac{f(a + h, b + k) - f(a,b) - h f_x(a,b) - k f_y(a,b)}{\sqrt{h^2 + k^2}} = 0$$

Consequently, $f(x,y)$ is differentiable at (a,b).

Some theorems of Differentiability of Real Valued Functions of Two Variables:

Theorem : If $f(x,y)$ and $g(x,y)$ are differentiable at (a,b) then their Sum, Difference, Product and Quotient are also differentiable at the point (a, b) i.e., :

(i) $f(x, y) \pm g(x, y)$ is differentiable   and $d(f \pm g) = df + dg$.

(ii) $f(x, y) g(x, y)$ is differentiable   and $d(fg) = fdg + gdf$.

(iii) $f(x, y) / g(x, y)$ is differentiable   when $g(x, y) \neq 0$

and $d(f/g) = (gdf - fdg) / g^2$

Condition for Differentiability in Polar Co-ordinates :

Let $f(x, y)$ be differentiable at (a, b), then

$$\lim_{(h,k) \to (0,0)} \frac{f(a + h, b + k) - f(a,b) - h f_x(a,b) - k f_y(a,b)}{\sqrt{h^2 + k^2}} = 0$$

Substituting $h = r \cos \theta$ and $k = r \sin \theta$ and taking limit $r \to 0$,

$$\lim_{r \to 0} \left[\frac{f(a + r \cos \theta, b + r \sin \theta) - f(a,b)}{r} - \cos \theta f_x(a,b) - \sin \theta f_y(a,b) \right] = 0$$

$$\Rightarrow \quad \lim_{r \to 0} \frac{f(a + r \cos \theta, b + r \sin \theta)}{r} = \cos \theta f_x(a,b) + \sin \theta f_y(a,b).$$

Ex. $f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & \text{when } (x,y) \neq (0,0) \\ 0 & , \text{ when } (x,y) = (0,0) \end{cases}$

Show that the function f is not differentiable at the origin.

Sol. $f_x(0,0) = \lim\limits_{h \to 0} \dfrac{\left(\dfrac{h^3 - 0}{h^2 + 0} - 0 \right)}{h} = 1$

$f_y(0,0) = \lim\limits_{k \to 0} \dfrac{\left(\dfrac{0 - k^3}{0 + k^2} - 0 \right)}{k} = -1$

Both the partial derivatives exists but are not equal. Therefore, f is not differentiable at (0, 0).

Ex. Show that the function f(x, y) defined by

$$f(x,y) = \begin{cases} x^2 \sin\dfrac{1}{x} + y^2 \sin\dfrac{1}{y}, & \text{when } x \neq 0, y \neq 0 \\ 0, & \text{when } x = 0, y = 0 \end{cases}$$

is differentiable at (0, 0).

Sol. By definition,

$$f_x(0,0) = \lim\limits_{h \to 0} \frac{f(0,0) - f(0,0)}{h} = \lim\limits_{h \to 0} \frac{h^2 \sin(1/h) - 0}{h} = \lim\limits_{h \to 0} h \sin\frac{1}{h} = 0$$

Similarly, $f_y(0, 0) = 0$

Now $h = r \cos\theta$ and $k = r \sin\theta$, then

$$\lim\limits_{r \to 0}\left[\frac{f(r\cos\theta, r\sin\theta) - f(0,0)}{r} - \cos\theta.f_x(0,0) - \sin\theta.f_y(0,0) \right]$$

$$= \lim\limits_{r \to 0}\left[\frac{r^2 \cos^2\theta \sin\left(\dfrac{1}{r\cos\theta}\right) + r^2 \sin^2\theta \sin\left(\dfrac{1}{r\sin\theta}\right)}{r} \right]$$

$[\because f(0,1) = 0, f_x(0,0) = 0 \text{ and } f_y(0,0) = 0]$

$$= \lim\limits_{r \to 0} r\left[\cos^2\theta \sin\left(\dfrac{1}{r\cos\theta}\right) + \sin^2\theta \sin\left(\dfrac{1}{r\sin\theta}\right) \right] = 0.$$

Hence the function f(x,y) is differential at (0,0).

Ex. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by setting

f(x, y) = $x^2 \sin(1/x) + y^2 \sin(1/y)$, when $xy \neq 0$,

f(x, 0) = $x^2 \sin(1/x)$, when $x \neq 0$,

f(0, y) = $y^2 \sin(1/y)$, when $y \neq 0$,

f(0, 0) = 0.

Sol. We have

$$f_x(x, y) = 2x \sin(1/x) - \cos(1/x), \quad x \neq 0, \qquad \qquad ...(1)$$
$$f_x(0, y) = 0.$$

Again $\quad f_y(x, y) = 2y \sin(1/y) - \sin(1/y), \quad y \neq 0,$
$$f_y(x, 0) = 0.$$

Since $\lim_{x \to 0} \cos(1/x)$ does not exist, so by (1), f_x is not continuous at (0, 0). Similarly f_y is not continuous at (0, 0).

Now $\quad f_x(0, 0) = \lim_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \dfrac{h^2 \sin(1/h) - 0}{h}.$

$$= \lim_{h \to 0} h \sin(1/h) = 0.$$

Similarly $f_y(0, 0) = 0$. Thus f_x, f_y both exist at (0, 0) but are not continuous at (0, 0).
f is differentiable at (0, 0).
We shall verify that

$$f(h, k) - f(0, 0) = Ah + Bk + \sqrt{h^2 + k^2} \ g(h, k) \qquad \qquad ...(2)$$

where $\quad \lim_{(h,k) \to (0,0)} g(h, k) = 0$

We know $A = f_x(0, 0) = 0$, $B = f_y(0, 0) = 0$. Also $f(0, 0) = 0$. $\qquad \qquad ...(3)$
From (2) and (3), we obtain

$$h^2 \sin(1/h) + k^2 \sin(1/k) = \sqrt{h^2 + k^2} \ g(h, k)$$

or $\quad g(h, k) = \dfrac{h}{\sqrt{h^2 + k^2}} \cdot h \sin(1/h) + \dfrac{k}{\sqrt{h^2 + k^2}} \cdot k \sin(1/k)$

$\qquad \qquad < h \sin(1/h) + k \sin(1/k) \qquad \qquad \{\because \ h, k \leq \sqrt{h^2 + k^2}\ ]$

$\therefore \qquad \lim_{(h,k) \to (0,0)} g(h, k) = 0. \qquad \qquad [\because \lim_{h \to 0} h \sin(1/h) = 0 = \lim_{k \to 0} k \sin(1/k)]$

Hence f is differentiable at (0, 0).

Ex. Show that the function $f(x,y) = \sqrt{x^2 + y^2}$ is not differentiable at (0, 0).

Sol. If f were differetiable then the partial derivatives $\dfrac{\partial f}{\partial x}$ (0, 0) would exist. But that partial derivative is the limit

$$\lim_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \dfrac{\sqrt{h^2}}{h}$$

Now notice that $\dfrac{\sqrt{h^2}}{h}$ is +1 if h > 0 and −1 if h < 0. Therefore, the limit as h → 0 does not exist and so the partial derivative does not exist. Hence f(x,y) is not differentiable at (0, 0).

Ex. Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

is not continuous at (0, 0) (and therefore not differentiable there) even though $f_x(0,0)$ and $f_y(0,0)$ exist.

Sol. Continuity implies that

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$$

along any curve through (0, 0). If we approach the origin along the line y = x then :

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,x)\to 0} \frac{x \cdot x}{x^2 + x^2} = \frac{1}{2} \neq f(0,0).$$

Hence, f(x,y) is not continuous at (0,0). However, the partial derivatives do exist at (0,0) :

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} 0 = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} 0 = 0$$

Theorem : If $f(x,y)$, $f_x(x,y)$, and $f_y(x,y)$ are continuous for all (x, y) in the disk $(x - a)^2 + (y - b)^2 < \delta$, for some $\delta > 0$ then f(x, y) is differentiable at (a, b).

Ex. In what region is the function f(x,y) = $\sqrt{y^2 - x^2}$ differentiable?

Sol. Since $f_x(x,y) = -x(y^2 - x^2)^{-\frac{1}{2}}$ and $f_y(x,y) = y(y^2 - x^2)^{-\frac{1}{2}}$, according to the above theorem the function is differentiable everywhere in the region above y = |x| and below y = −|x| as shown in Fig.
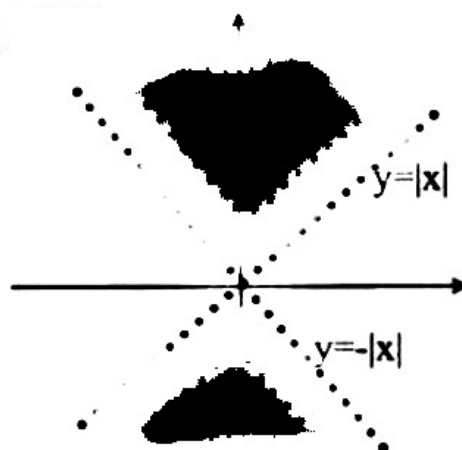
6. MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

Reminder

For a function of one variable f(x), we find the local maxima/minima by differentiation. Maxima/minima occur when f'(x) = 0.

- x = a is a maximum if f'(a) = 0 and f"(a) < 0;
- x = a is a minimum if f'(a) = 0 and f"(a) > 0;

A point where f"(a) = 0 and f"'(a) ≠ 0 is called a point of inflection. Geometrically, the equation y = f(x) represents a curve in the two-dimensional (x,y) plane, and we call this curve the graph of the function f(x).

Functions of two variables

Our aim is to generalise these ideas to functions of two variables. Such a function would be written as

$$z = f(x,y)$$

where x and y are the independent variables and z is the dependent variable. The graph of such a function is a surface in three dimensional space. A simple example might be

$$z = \frac{1}{1 + x^2 + y^2}.$$

z is the height of the surface above a point (x,y) in the X-Y plane.

For functions z = f(x,y) the graph (i.e. the surface) may have maximum points or minimum points (or both). But for surfaces there is a third possibility - a saddle point.

A point (a,b) which is a maximum, minimum or saddle point is called a stationary point. The actual value at a stationary point is called the stationary value. What we need is a mathematical method for finding the stationary points of a function f(x,y) and classifying them into maximum, minimum or saddle point. This method is analogous to, but more complicated than, the method of working out first and second derivatives for functions of one variable.

Let's remind about partial derivatives. The sort of function we have in mind might be something like.

$$f(x,y) = x^2y^3 + 3y + x$$

and the partial derivatives of this would be

$$\frac{\partial f}{\partial x} = 2xy^3 + 1$$

$$\frac{\partial f}{\partial y} = 3x^2y^2 + 3$$

$$\frac{\partial^2 f}{\partial x^2} = 2y^3$$

$$\frac{\partial^2 f}{\partial y^2} = 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6xy^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = 6xy^2, \text{ same as } \frac{\partial^2 f}{\partial x \partial y}$$

Note that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

This is true for any well behaved function. In terms of notation, we will frequently use the other, subscript, notation for partial derivatives:

$$f_x = \frac{\partial f}{\partial x}, \; f_y = \frac{\partial f}{\partial y},$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \; f_{yy} = \frac{\partial^2 f}{\partial y^2}, \; f_{xy} = \frac{\partial^2 f}{\partial y \partial x} \text{ or } \frac{\partial^2 f}{\partial x \partial y}$$

Finding stationary points

To find the stationary points of $f(x,y)$, work out $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and set both to zero. This gives you two equations for two unknowns x and y. Solve these equations for x and y (often there is more than one solution, as indeed you should expect. After all, even functions of one variable may have both maximum and minimum points).

Classifying stationary points

The procedure for classifying stationary points of a function of two variables is analogous to, but somewhat more involved, than the corresponding 'second derivative test' for functions of one variable. Below is, essentially, the second derivative test for functions of two variables:

Let (a,b) be a stationary point, so that $f_x = 0$ and $f_y = 0$ at (a, b). Then:

- if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a,b) then (a,b) is a saddle point.

- if $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) then (a,b) is either a maximum or a minimum.

    Distinguish between these as follows:
    - if $f_{xx} < 0$ and $f_{yy} < 0$ at (a,b) then (a,b) is a maximum point
    - if $f_{xx} > 0$ and $f_{yy} > 0$ at (a,b) then (a,b) is a minimum point

If $f_{xx}f_{yy} - f_{xy}^2 = 0$ then anything is possible. More advanced methods are required to classify the stationary point properly.

Let's give some idea where the above conditions come from. It is all based on Taylor's theorem for a function of two variables. Taylor's theorem for a function of one variable is

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots$$

For a function of two variables Taylor's theorem is

$$f(a+h,b+k) = f(a,b) + h\frac{\partial f}{\partial x}(a,b) + k\frac{\partial f}{\partial y}(a,b) + \frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2hk\frac{\partial^2 f}{\partial x \partial y}(a,b) + k^2\frac{\partial^2 f}{\partial y^2}(a,b) \right]$$

+ higher order (and more complicated) terms

The higher order terms can be neglected in straightforward cases. Lets suppose (a,b) is a maximum point. Then $f_x = 0$ and $f_y = 0$ at (a,b) and, because (a,b) is a local maximum the function must be smaller at neighbouring points, i.e. when h and k are sufficiently small,

$$f(a + h, b + k) < f(a, b)$$

But from Taylors theorem, neglecting higher order terms and noting that the first derivative terms are zero at (a,b),

$$f(a + h, b + k) = f(a,b) + \frac{1}{2}[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}] \qquad \text{at } (a,b).$$

However $f(a + h, b + k) < f(a, b)$, hence

$$h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy} < 0 \qquad \text{at } (a,b)$$

for all small values of h and k. Dividing by k^2 gives

$$\left(\frac{h}{k}\right)^2 f_{xx} + 2\left(\frac{h}{k}\right)f_{xy} + f_{yy} < 0.$$

Let $\xi = h/k$. Then even though h and k are both small, ξ doesn't have to be small.

So we have

$$f_{xx}\xi^2 + 2f_{xy}\xi + f_{yy} < 0 \qquad \text{for all real numbers } \xi.$$

Thus we have a quadratic expression that is negative for all values of its variable ξ (and so, in particular, has no roots). A few graphs will show that this is only possible if $f_{xx} < 0$, $f_{yy} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ - the latter condition is the one to do with having no roots. All these inequalities hold at (a,b).

Similar analysis yields the conditions under which a stationary point is a minimum or saddle point.

Definition

1. A function f(x, y) is said to have a maximum at a point (a, b) if there exists a nbd. N of (a, b) such that

    $$f(a + h, b + k) < f(a, b) \quad \forall \ (a + h, b + k) \in N \sim \{(a, b)\}.$$

    The set N ~ {a, b)} is called a deleted nbd. of (a, b).

2. A function f(x, y) is said to have a minimum at a point (a, b) if there exists a nbd. N of (a, b) such that

    $$f(a + h, b + k) > f(a, b) \quad \forall \ (a + h, b + k) \in N \sim \{(a, b)\}.$$

3. A function f(x, y) is said to have an extremum at a point (a, b) if it has either a maximum or minimum at (a, b).

Theorem : If a function f(x, y) has an extremum at any point (a, b) of the domain of f and if f admits of partial derivatives f_x and f_y at (a, b), then

$$f_x(a,b) = 0 \text{ and } f_y(a, b) = 0.$$

Proof. Suppose f(x, y) has a maximum at (a, b). Then the function f(a, b) of a single variable x must have a maximum at x = a and so

$$f_x(a, b) = 0 \text{ when } x = a \text{ i.e., } f_x(a, b) = 0.$$

Again the function f(a, y) of a single variable y must have a maximum at y = b and so

$$f_y(a, y) = 0 \text{ when } y = b \text{ i.e., } f_y(a, b) = 0.$$

Exactly similar is the case when f has a minimum at (a, b).

4. A point (a, b) is said to be a critical point for a function f(x, y) if f_x and f_y both exist at (a, b) and

$f_x(a, b) = 0, f_y(a, b) = 0.$

If (a, b) is a critical point for a function f(x, y), then f(a, b) is called a stationary value of f.

Ex. Find the maxima and minima of the function

$f(x, y) = x^3 + y^3 - 3x - 12y + 20.$

Sol. We have

$f_x = 3x^2 - 3, f_y = 3y^2 - 12, f_{xx} = 6x, f_{yy} = 6y, f_{xy} = 0.$

To determine the critical points, we solve

$f_x = 0, f_y = 0 \Rightarrow 3x^2 - 3 = 0, 3y^2 - 12 = 0$

$\Rightarrow$ $x = \pm 1, y = \pm 2.$ Hence the critical points are

(1, 2), (1 – 2), (–1, 2), (–1, –2).

At (1, 2),

$A = f_{xx} = 6 > 0, B = 0, C = 12.$

$AC - B^2 = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 6 \times 12 - 0 = 72 > 0.$

Thus f has minimum at (1, 2).

At (1, –2),

$AC - B^2 = 6 \times (-12) - 0 < 0.$

Thus f has neither maximum nor minimum at (1, –2).

At (–1, 2),

$AC - B^2 = -6 \times 12 - 0 < 0.$

Thus f has neither maximum nor minimum at (–1, 2).

At (–1, –2), A = –6 < 0,

$AC - B^2 = (-6)(-12) - 0 > 0.$

Thus f has maximum at (–1, –2).

Example. Lets work out the stationary points for the function

$f(x,y) = x^2 + y^2$

and classify them into maxima, minima and saddles.

We need all the first and second derivatives so lets work them out. we have

$f_x = 2x$

$f_y = 2y$

$f_{xx} = 2$

$f_{yy} = 2$

$f_{xy} = 0$

For stationary points we need $f_x = f_y = 0.$ This gives 2x = 0 and 2y = 0 so that there is just one stationary point, namely (x,y) = (0,0). We now need to classify it.

Now

$f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - 0^2 = 4 > 0$

so it is either a max or a min. But $f_{xx} = 2 > 0$ and $f_{yy} = 2 > 0.$ Hence it is a minimum. Our conclusion is that this function has just one stationary point (0, 0) and that it is a minimum.

Example. $f(x, y) = e^{-(x^2+y^2)}$

The first and second order partial derivatives of this function are :

$f_x = -2xe^{-(x^2+y^2)}$

$$f_y = -2ye^{-(x^2+y^2)}$$

$$f_{xx} = -2e^{-(x^2+y^2)}(1-2x^2) \qquad \text{by the product rule}$$

$$f_{yy} = -2e^{-(x^2+y^2)}(1-2y^2)$$

$$f_{xy} = 4xye^{-(x^2+y^2)}$$

Stationary points are when $f_x = 0$ and $f_y = 0$ and so there is only one stationary point, at $(x,y) =$ (0, 0). Substituting $(x,y) = (0,0)$ into the expressions for f_{xx}, f_{yy} and f_{xy} gives

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 0$$

Therefore

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - 0^2 = 4 > 0$$

So that $(0,0)$ is either a min or a max. Since $f_{xx} < 0$ and $f_{yy} < 0$ it is a maximum.

Example. $f(x,y) = 2 - x^2 - xy - y^2$

For this function

$$f_x = -2x - y$$
$$f_y = -x - 2y$$
$$f_{xx} = -2$$
$$f_{yy} = -2$$
$$f_{xy} = -1$$

For stationary points, $-2x - y = 0$ and $-x - 2y = 0$ so again the only possibility is $(x, y) = (0,0)$. We have

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - (-1)^2 = 3 > 0$$

so that $(0, 0)$ is either a max or a min. Since $f_{xx} < 0$ and $f_{yy} < 0$, So it is a maximum.

7. FUNCTIONS OF THREE VARIABLES

We extend our study of multivariable functions to functions of three variables.

Definition. Function of Three Variables. Let D be a subset of $\mathbb{R}^3$. A **function f of three variables** is a rule that assigns each triple (x, y, z) in D a value w = f(x, y, z) in $\mathbb{R}$. D is the **domain** of f; the set of all outputs of f is the **range**.

Note how this definition closely resembles that of **Function of two variables**.

Ex. Let f(x, y, z) = $\dfrac{x^2 + z + 3\sin(y)}{x + 2y - z}$. Evaluate f at the point (3, 0, 2) and find the domain and range of f.

Sol. f(3, 0, 2) = $\dfrac{3^2 + 2 + 3\sin(0)}{3 + 2(0) - 2}$ = 11

As the domain of f is not specified, we take it to be the set of all triples (x, y, z) for which f(x, y, z) is defined. As we cannot divide by 0, we find the domain D is

$$D = \{(x, y, z) \mid x + 2y - z \neq 0\}.$$

We recognize that the set of all points in $\mathbb{R}^3$ that are not in D form a plane in space that passes through the origin (with normal vector $\langle 1, 2, -1 \rangle$).

We determine the range R is $\mathbb{R}$; that is, all real numbers are possible outputs of f. There is no set way of establishing this. Rather, to get numbers near 0 we can let y = 0 and choose z ≈ – x^2. To get numbers of arbitrarily large magnitude, we can let z ≈ x + 2y.

Level Surfaces

If is very difficult to produce a meaningful graph of a function of three variables. A function of one variable is a curve drawn in 2 dimensions; a function of two variables is a surface drawn in 3 dimensions; a function of three variables is a hypersurface drawn in 4 dimensions.

There are a few techniques one can employ to try to "picture" a graph of three variables. One is an analogue to level curves: level surfaces. Given w = f(x, y, z), the level surface at w = c is the surface in space formed by all points (x, y, z) where f(x, y, z) = c.

Ex. **Finding level surfaces.** If a point source S is radiating energy, the intensity I at a given point P in space is inversely proportional to the square of the distance between S and P. That is, when S = (0, 0, 0).

I(x, y, z) = $\dfrac{k}{x^2 + y^2 + z^2}$ for some constant k.

Let k = 1; find the level surfaces of I.

Sol. If energy (say, in the form of light) is emanating from the origin, its intensity will be the same at all points equidistant from the origin. That is, at any point on the surface of a sphere centered at the origin, the intensity should be the same. Therefore, the level surfaces are spheres.

We now find this mathematically. The level surface at I = c is defined by

$$c = \frac{1}{x^2 + y^2 + z^2}$$

A small amount of algebra reveals

$$x^2 + y^2 + z^2 = \frac{1}{c}$$

Given an intensity c, the level surface I = c is a sphere of radius $1/\sqrt{c}$, centered at the origin.

Definition

Open Balls, Limit, Continuous

1. An **open ball** in $\mathbb{R}^3$ centered at (x_0, y_0, z_0) with radius r is the set of all points (x, y, z) such that $\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$ = r.

2. Let D be an open set in $\mathbb{R}^3$ containing (x_0, y_0, z_0), and let f(x, y, z) be a function of three variables defined on D, except possibly at (x_0, y_0, z_0). The **limit** of f(x, y, z) as (x, y, z) approaches (x_0, y_0, z_0) is L, denoted

$$\lim_{(x,y,z) \to (x_0,y_0,z_0)} f(x,y,z) = L,$$

means that given any $\varepsilon > 0$, there is a $\delta > 0$ such that for all $(x, y, z) \neq (x_0, y_0, z_0)$, if (x, y, z) is in the open ball centered at (x_0, y_0, z_0) with radius δ, then $|f(x, y, z) - L| < \varepsilon$.

3. Let f(x, y, z) be defined on an open ball B containing (x_0, y_0, z_0). f is **continuous** at (x_0, y_0, z_0) if $\lim_{(x,y,z) \to (x_0,y_0,z_0)} f(x,y,z) = f(x_0,y_0,z_0)$.

These definitions can also be extended naturally to apply to functions of four or more variables. Theorem also applies to function of three or more variables, allowing us to say that the function

$$f(x,y,z) = \frac{e^{x^2+y}\sqrt{y^2 + z^2 + 3}}{\sin(xyz) + 5}$$

is continuous everywhere.

Partial Derivatives and Functions of Three Variables

The concepts underlying partial derivatives can be easily extend to more than two variables. We give some definitions and examples in the case of three variables and one can extend these definitions to more variables if needed.

Definition : Partial Derivatives with Three Variables. Let w = f(x, y, z) be a continuous function on an open set S in $\mathbb{R}^3$.

The **partial derivative of f with respect to x** is :

$$f_x(x,y,z) = \lim_{h \to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

Similar definitions hold for $f_y(x, y, z)$ and $f_z(x, y, z)$.

By taking partial derivatives of partial derivatives, we can find second partial derivatives of f with respect to z then y, for instance, just as before.

Ex. Partial derivatives of functions of three variables.

For each of the following, find f_x, f_y, f_z, f_{xz}, f_{yz}, and f_{zz}.

1. $f(x, y, z) = x^2y^3z^4 + x^2y^2 + x^3z^3 + y^4z^4$

2. $f(x, y, z) = x \sin(yz)$

Sol. 1. $f_x = 2xy^3z^4 + 2xy^2 + 3x^2z^3$;

$f_y = 3x^2y^2z^4 + 2x^2y + 4y^3z^4$;

$f_z = 4x^2y^3z^3 + 3x^3z^2 + 4y^4z^3$;

$f_{xz} = 8xy^3z^3 + 9x^2z^2$;

$f_{yz} = 12x^2y^2z^3 + 16y^3z^3$;

$f_{zz} = 12x^2y^3z^2 + 6x^3z + 12y^4z^2$

2. $f_x = \sin(yz)$;

    $f_y = xz \cos(yz)$;

    $f_z = xy \cos(yz)$;

    $f_{xz} = y \cos(yz)$;

    $f_{yz} = x \cos(yz) - xyz \sin(yz)$;

    $f_{zz} = -xy^2 \sin(xy)$

Higher Order Partial Derivatives

We can continue taking partial derivatives of partial derivatives of partial derivatives of ...; we do not have to stop with second partial derivatives.

We do not formally define each higher order derivative, but rather give just a few examples of the notation.

$$f_{xyz}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) \text{ and } f_{xyz}(x,y,z) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)$$

Ex.   Higher order partial derivatives.

1. Let $f(x, y) = x^2y^2 + \sin(xy)$. Find f_{xxy} and f_{yxx}.

2. Let $f(x, y, z) = x^3e^{xy} + \cos(z)$. Find f_{xyz}.

Sol.   1. To find f_{xxy}, we first find f_x, then f_{xx}, then f_{xxy}:

$f_x = 2xy^2 + y \cos(xy)$

$f_{xx} = 2y^2 - y^2 \sin(xy)$

$f_{xxy} = 4y - 2y \sin(xy) - xy^2 \cos(xy)$.

To find f_{yxx}, we first find f_y, then f_{yx}, then f_{yxx}:

$f_y = 2x^2y + x \cos(xy)$

$f_{yx} = 4xy + \cos(xy) - xy \sin(xy)$

$f_{yxx} = 4y - y \sin(xy) - (y \sin(xy) + xy^2 \cos(xy))$

$\qquad = 4y - 2y \sin(xy) - xy^2 \cos(xy)$.

Note how $f_{xxy} = f_{yxx}$.

2. To find f_{xyz}, we find f_x, then f_{xy}, then f_{xyz}:

$f_x = 3x^2 e^{xy} + x^3 ye^{xy}$

$f_{xy} = 3x^3 e^{xy} + x^3 e^{xy} + x^4 ye^{xy} = 4x^3 e^{xy} + x^4 ye^{xy}$

$f_{xyz} = 0$.

In the previous example we saw that $f_{xxy} = f_{yxx}$; this is not a coincidence. While we do not state this as a formal theorem, as long as each partial derivative is continuous, it does not matter the order in which the partial derivatives are taken. For instance, $f_{xxy} = f_{xyx} = f_{yxx}$.

Differentiability of Functions of Three Variables

The definition of differentiability for functions of three variables is very similar to that of functions of two variables. We again start with the total differential.

Definition : Total Differential. Let $w = f(x, y, z)$ be continuous on an open set S. Let dx, dy and dz represent changes in x, y and z, respectively. Where the partial derivatives f_x, f_y and f_z exist, the *total differential of w* is

$$dz = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz$$

This differential can be a good approximation of the change in w when $w = f(x, y, z)$ is differentiable.

Definition : Multivariable Differentiability. Let $w = f(x, y, z)$ be defined on an open ball B containing (x_0, y_0, z_0) where $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$ and $f_z(x_0, y_0, z_0)$ exist. Let dw be the total differential of w at (x_0, y_0, z_0). Let $\Delta w = f(x_0 + dx, y_0 + dy, z_0 + dz) - f(x_0, y_0, z_0)$, and let E_x, E_y and E_z be functions

of dx, dy and dz such that

$$\Delta w = dw = E_x dx + E_y dy + E_z dz$$

1. f is differentiable at (x_0, y_0, z_0) if, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $\|\langle dx, dy, dz\rangle\| < \delta$, then $\|\langle E_x, E_y, E_z\rangle\| < \varepsilon$.

2. f is differentiable on B if f is differentiable at every point in B. If f is differentiable on $\mathbb{R}^3$, we say that f is differentiable everywhere.

Just as before, this definition gives a rigorous statement about what it means to be differentiable that is not very intuitive. We follow it with a theorem similar to Theorem.

Theorem : Continuity and Differentiability of Functions of Three Variables. Let w = f(x, y, z) be defined on an open ball B containing (x_0, y_0, z_0).

1. If f is differentiable at (x_0, y_0, z_0), then f is continuous at (x_0, y_0, z_0).

2. If f_x, f_y and f_z are continuous on B, then f is differentiable on B.

This set of definition and theorem extends to functions of any number of variables. The theorem again gives us a simple way of verifying that most functions that we encounter are differentiable on their natural domains.

This section has given us a formal definition of what it means for a functions to be "differentiable," along with a theorem that gives a more accessible understanding. The following sections return to notions prompted by our study of partial derivatives that make use of the fact that most functions we encounter are differentiable.

Maxima & Minima of Three Variables Functions

Let u = f(x, y, z) be a function of three variables x, y and z, then we can find the maximum and minimum value of the function as like the two variables.

Suppose f(x, y, z) is a given function of three independent variables x, y and z.

- Find $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$.

- Now solve the equations $\dfrac{\partial f}{\partial x} = 0, \dfrac{\partial f}{\partial y} = 0$ and $\dfrac{\partial f}{dz} = 0$.

- After solving we get the points at which the function f(x, y, z) may be maximum or a minimum that are

$$(a_1, b_1, c_1), (a_2, b_2, c_2), \dots$$

- To discuss the maximum or minimum value of f(x, y, z) at any point (a_1, b_1, c_1), find six partial derivative at this point.

$$A = \left(\frac{\partial^2 f}{\partial x^2}\right)_{\substack{x=a_1 \\ y=b_1 \\ z=c_1}}, B = \left(\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a_1 \\ y=b_1 \\ z=c_1}}, C = \left(\frac{\partial^2 f}{\partial z^2}\right)_{\substack{x=a_1 \\ y=b_1 \\ z=c_1}}$$

$$F = \left(\frac{\partial^2 f}{\partial y \partial z}\right)_{\substack{x=a_1 \\ y=b_1 \\ z=c_1}}, G = \left(\frac{\partial^2 f}{\partial z \partial x}\right)_{\substack{x=a_1 \\ y=b_1 \\ z=c_1}}, H = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\substack{x=a_1 \\ y=b_1 \\ z=c_1}}$$

- Find the expression

$$A, \quad A_1 = \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \quad A_2 = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} \qquad \dots(1)$$

- If all A, A_1 and A_2 are positive, then f(x, y, z) is minimum at (a_1, b_1, c_1).
- If A, A_1 and A_2 are alternately negative and positive, then f(x, y, z) is maximum at (a_1, b_1, c_1).
- If above conditions not satisfied, then f(x, y, z) is neither maximum nor minimum at (a_1, b_1, c_1).

Example.   Discuss the maximum and minimum value of

$$u = x^2 + y^2 + z^2 + x - 2z - xy.$$

As given $u = x^2 + y^2 + z^2 + x - 2z - xy$, then

$$u_x = 2x - y + 1 = 0, \quad u_y = -x + 2y = 0, \quad u_z = 2z - 2 = 0$$

On solving these equations we get x = –2/3, y = –1/3, z = 1. Therefore $\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$ is only critical point where function will be either maximum or minimum. Now we will decide it.

Now the values of following derivatives

$$A = 2, B = 2, C = 2, F = 0, G = 0, H = -1$$

Now we have

$$A = 2, \quad A_1 = \begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

$$A_2 = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6$$

Since all these A, A_1 and A_2 are positive, we have a minimum of u at the point $\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$.

Example.   Show that the point such that the sum of the sum of the squares of its distances from n given points shall be minimum, is the center of the mean position of the given points.

Let the n given points be (a_1, b_1, c_1), (a_2, b_2, c_3), ... (a_n, b_n, c_n) and let (x, y, z) be the coordinate of the required point.

If u denotes the sum of the squares of the distances of (x, y, z) from the n given points, then

$$u = \sum \left[(x - a_i)^2 + (y - b_i)^2 + (z - c_i)^2\right]$$

$$= \sum (x - a_i)^2 + \sum (y - b_i)^2 + \sum (z - c_i)^2$$

For maximum or a minimum of u, we must have

$$u_x = 2\sum (x - a_i) = 2nx - 2\sum a_i = 0 \qquad \qquad ...(2)$$

$$u_y = 2\sum (y - b_i) = 2ny - 2\sum b_i = 0 \qquad \qquad ...(3)$$

$$u_z = 2\sum (z - c_i) = 2nz - 2\sum c_i = 0 \qquad \qquad ...(4)$$

Solving these above equations, we get

$$x = \frac{\sum a_i}{n}, \quad y = \frac{\sum b_i}{n}, \quad z = \frac{\sum c_i}{n}$$

Now $A = 2n, B = 2n, C = 2n, F = G = H = 0$

We have $A = 2n_1$, $A_1 = 4n^2$, $A_2 = 8n^3$ Since there three expression are all positive, u is minimum when

$$x = \frac{\sum a_i}{n}, \ y = \frac{\sum b_i}{n}, \ z = \frac{\sum c_i}{n}$$

So u is minimum when the point (x, y, z) is the centre of the mean position of the n given points.

Ex. Find the range of $f(x, y, z) = \dfrac{x}{x + 2y - 4z}$

Sol. $x + 2y - 4z \neq 0$; the set of points of $\mathbb{R}^3$ NOT in the domain form a plane through the origin. Range: $\mathbb{R}$

Ex. Find the range of $f(x, y, z) = \dfrac{1}{1 - x^2 - y^2 - z^2}$

Sol. $x^2 + y^2 + z^2 \neq 1$; the set of points in $\mathbb{R}^3$ NOT in the domain form a sphere of radius 1. Range: $(-\infty, 0) \cup [1, \infty)$

Ex. Find the range of $f(x, y, z) = \sqrt{z - x^2 + y^2}$

Sol. $z \geq x^2 - y^2$; the set of points in $\mathbb{R}^3$ above (and including) the hyperbolic paraboloid $z = x^2 - y^2$. Range: $[0, \infty)$

8. METHOD OF LAGRANGE MULTIPLIER

The method of Lagrange multipliers allows us to maximize or minimize functions with the constraint that we only consider points on a certain surface. To find critical points of a function $f(x, y, z)$ on a level surface $g(x, y, z) = C$ (or subject to the constraint $g(x, y, z) = C$), we must solve the following system of simultaneous equations:

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

$$g(x, y, z) = C$$

Remembering that ∇f and ∇g are vectors, we can write this as a collection of four equations in the four unknowns x, y, z, and λ:

$$f_x(x, y, z) = \lambda g_x(x, y, z)$$
$$f_y(x, y, z) = \lambda g_y(x, y, z)$$
$$f_z(x, y, z) = \lambda g_z(x, y, z)$$
$$g(x, y, z) = C$$

The variable λ is a dummy variable called a "Lagrange multiplier"; we only really care about the values of x, y, and z.

Once you have found all the critical points, you plug them into f to see where the maxima and minima of the function occurs. The critical points where f is greatest are called points of maxima and the critical points where f is smallest are called points of minima.

Ex. Use the method of Lagrange multipliers to maximize

$$f(x, y) = 3xy$$

subject to the constraint $x + y = 8$.

Sol. First let $f(x, y) = 3xy$ and $g(x, y) = x + y - 8$.

then defined a new function F as

$$F(x, y, z) = f(x, y) - \lambda g(x, y)$$
$$= 3xy - \lambda(x + y - 8)$$

To find the critical numbers of F, begin by finding the partial derivatives of F with respect to x, y and λ. Then, set of partial derivatives equal to zero.

$F_x(x, y, \lambda) = 3y - \lambda$	$\Rightarrow$	$3y - \lambda = 0$	...(1)
$F_y(x, y, \lambda) = 3x - \lambda$	$\Rightarrow$	$3x - \lambda = 0$	...(2)
$F_\lambda(x, y, \lambda) = -x - y + 8$	$\Rightarrow$	$-x - y + 8 = 0$	...(3)

Solving for λ in the first equation (1) gives

$$3y - \lambda = 0 \Rightarrow \lambda = 3y$$

Substituting for λ in the second (2) equation

$$3x - 3y = 0 \Rightarrow 3x - 3y \quad \text{or} \quad x = y$$

Now, substitute the value of x in equation (3).

$$-y - y + 8 = 0$$
$$2y = 8$$
$$y = 4$$

Using this value, we can conclude that the critical values are $x = 4$, $y = 4$. Which implies that the maximum is

$$f(x, y) = 3xy$$
$$= 3(4)(4) = 48.$$

Ex. Find the maximum of

$$V = xyz$$

subject to the constraint $6x + 4y + 3z - 24 = 0$

Sol. First, let $f(x, y, z) = xyz$ and $g(x, y, z) = 6x + 4y + 3z - 24$. Then, define a new function F as

$$F(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)$$
$$= xyz - \lambda(6x + 4y + 3z - 24).$$

To find the critical numbers of F, begin by finding the partial derivatives of F with respect to x, y, z, and λ. Then, set the partial derivatives equal to zero.

$$F_x(x, y, z, \lambda) = yz - 6\lambda \quad \Rightarrow \quad yz - 6\lambda = 0$$
$$F_y(x, y, z, \lambda) = xz - 4\lambda \quad \Rightarrow \quad xz - 4\lambda = 0$$
$$F_z(x, y, z, \lambda) = xy - 3\lambda \quad \Rightarrow \quad xy - 3\lambda = 0$$
$$F_\lambda(x, y, z, \lambda) = -6x - 4y - 3z + 24 \quad \Rightarrow \quad -6x - 4y - 3z + 24 = 0$$

Solving for λ in the first equation produces

$$yz - 6\lambda = 0 \quad \Rightarrow \quad \lambda = \frac{yz}{6}.$$

Substituting for λ in the second and third equations produces the following.

$$xz - 4\left(\frac{yz}{6}\right) = 0 \quad \Rightarrow \quad y = \frac{3}{2}x$$
$$xy - 3\left(\frac{yz}{6}\right) = 0 \quad \Rightarrow \quad z = 2x$$

Next, substituting for y and z in the equation $F_x(x, y, z, \lambda) = 0$ and solve for x.

$$F_x(x, y, z, \lambda) = 0$$
$$-6x - 4y - 3z + 24 = 0$$
$$-6x - 4\left(\frac{3}{2}x\right) - 3(2x) + 24 = 0$$
$$-18x = -24$$
$$x = \frac{4}{3}$$

Using this x-value, you can conclude that the critical values are $x = \frac{4}{3}$, $y = 2$, and $z = \frac{8}{3}$, which implies that the maximum is

$$V = xyz \qquad \text{Write objective function}$$
$$= \left(\frac{4}{3}\right)(2)\left(\frac{8}{3}\right) \qquad \text{Substitute value of x, y, and z.}$$
$$= \frac{64}{9} \text{ cubic units.} \qquad \text{Maximum volume.}$$

Ex. Minimize: $f(x, y) = x^2 + y^2$

Subject to : $2x + 6y = 2000$

Sol. $f(x, y) = x^2 + y^2$

$g(x, y) = 2x + 6y - 2000 = 0.$

$L(x, y, \lambda) = x^2 + y^2 - \lambda(2x + 6y - 2000)$

$$L_x(x, y, \lambda) = 2x - 2\lambda \quad \Rightarrow \quad 2x - 2\lambda = 0 \quad \dots(1)$$
$$L_y(x, y, \lambda) = 2y - 6\lambda \quad \Rightarrow \quad 2y - 6\lambda = 0 \quad \dots(2)$$
$$L_\lambda(x, y, \lambda) = -2x - 6y + 2000 \quad \Rightarrow \quad -2x - 6y + 2000 = 0 \quad \dots(3)$$

Solving for λ

$$2x - 2\lambda = 0 \quad \Rightarrow \quad \lambda = x$$

Putting value of λ in (2).

$$2y + 6x = 0 \quad \Rightarrow \quad 6x = 2y$$
$$y = 3x$$

Now Substituting y in (3)

$$-2x - 18x + 2000 = 0$$
$$\Rightarrow \quad -20x + 2000 = 0$$
$$x = 100$$

Using the value of x we can conclude that the critical points are x = 100, y = 300

Therefore the min value in $f(100, 300) = (100)^2 + (300)^2 = 10000 + 90000 = 1,00,000$

Ex. Find the maximum and minimum values of $f(x, y) = 81x^2 + y^2$ subject to the constraint $4x^2 + y^2 = 9$.

Sol. Let $\quad f(x, y) = 81 x^2 + y^2$

$$g(x, y) = 4x^2 + y^2 - 9$$

Then define a new function F as

$$F = f(x, y) - \lambda g(x, y)$$
$$= 81x^2 + y^2 - \lambda(4x^2 + y^2 - 9)$$
$$F_x(x, y, \lambda) = 162x - 8\lambda x \quad \Rightarrow \quad 162x - 8\lambda x = 0 \quad \dots(1)$$
$$F_y(x, y, \lambda) = 2y - 2y\lambda \quad \Rightarrow \quad 2y - 2y\lambda = 0 \quad \dots(2)$$
$$F_\lambda(x, y, \lambda) = -4x^2 - y^2 + 9 \quad \Rightarrow \quad -4x^2 - y^2 + 9 = 0 \quad \dots(3)$$

Solve for λ Let's start with the second one.

$$2y = 2y\lambda$$
$$\Rightarrow \quad \lambda = 1 \quad \text{or} \quad y = 0$$

Let $y = 0$, then in this case

$$4x^2 = 9 \quad \Rightarrow \quad x = \pm\frac{3}{2}$$

Therefore we get two points that are absolute extrema $\left(-\frac{3}{2}, 0\right)$ and $\left(\frac{3}{2}, 0\right)$.

Now Let $\lambda = 1$, in this case equation (1) reduces to

$$162x = 8x \quad \Rightarrow \quad 154x = 0 \quad \Rightarrow \quad x = 0$$

under this assumption we must have x = 0. Now substitute in (3) we get

$$y^2 = 9 \quad \Rightarrow \quad y = \pm 3$$

So this gives us two more points that are absolute extreme

(0, −3) and (0, 3)

Now, we have four points that are absolute extrema to determine the absolute extrema we need to evaluate the function at each of these points

$$f\left(-\frac{3}{2}, 0\right) = \frac{729}{4}, \quad f\left(\frac{3}{2}, 0\right) = \frac{729}{4}, \quad f(0, -3) = 9 \quad f(0, 3) = 9$$

The absolute maximum is $\dfrac{729}{4}$ = 182·25 which occurs at $\left(-\dfrac{3}{2}, 0\right)$ and $\left(\dfrac{3}{2}, 0\right)$.

The absolute minimum is which occurs at (0, –3) and (0, 3).

Ex. Find the maximum values of $f(x, y) = 8x^2 - 2y$ subject to the constraint $x^2 + y^2 = 1$.

Sol. Given $f(x, y) = 8x^2 - 2y$

$g(x, y) = x^2 + y^2 - 1$

$F(x, y, \lambda) = f(x, y) - \lambda g(x, y) = 8x^2 - 2y - \lambda(x^2 + y^2 - l)$

$F_x = 16x - 2\lambda x \qquad \Rightarrow \qquad 16x - 2\lambda x = 0 \qquad \qquad \text{...(1)}$

$F_y = -2 - 2\lambda y \qquad \Rightarrow \qquad -2 - 2\lambda y = 0 \qquad \qquad \text{...(2)}$

$F_\lambda = -x^2 - y^2 + 1 \qquad \Rightarrow \qquad -x^2 - y^2 + 1 = 0$

Solve for λ

by (1) $16x - 2\lambda x = 0$

$\Rightarrow \qquad x = 0 \quad \text{or} \quad \lambda = 8$

Let $x = 0$ then by (3), we get

$y^2 = 1 \Rightarrow y = \pm 1$

Therefore, we get two points that are absolute extrema (0, –1) and (0, 1)

Now assume $\lambda = 8$. In this case, we can plug the value in equation (2) we get.

$$-2 = 16y \Rightarrow y = -\dfrac{1}{8}$$

we now plug the value of y in equation (3)

$$x^2 + \dfrac{1}{64} = 1 \Rightarrow x = \pm\dfrac{3\sqrt{7}}{8}$$

this gives two more points that are absolute extrema.

Now $f\left(-\dfrac{3\sqrt{7}}{8}, -\dfrac{1}{8}\right) = \dfrac{65}{8}$, $f\left(\dfrac{3\sqrt{7}}{8}, -\dfrac{1}{8}\right) = \dfrac{65}{8}$

$f(0, -1) = 2$, $f(0, 1) = -2$

The absolute maximum is $\dfrac{65}{8}$ = 8.125 occurs at $\left(-\dfrac{3\sqrt{7}}{8}, -\dfrac{1}{8}\right)$ and $\left(\dfrac{3\sqrt{7}}{8}, -\dfrac{1}{8}\right)$

Ex. Find the minimum values of $f(x, y, z) = xyz$ subject to the constraint $x + 9y^2 + z^2 = 4$. Assume that $x \geq 0$.

Sol. $f(x, y, z) = xyz$

$g(x, y, z) = x + 9y^2 + z^2 = 4, \ x \geq 0$

$F(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)$

$= xyz - \lambda(x + 9y^2 + z^2 - 4)$

$F_x = yz - \lambda \qquad \Rightarrow \qquad yz - \lambda = 0 \qquad \qquad \text{...(1)}$

$F_y = xz - 18\lambda y \qquad \Rightarrow \qquad xz - 18\lambda y = 0 \qquad \qquad \text{...(2)}$

$F_z = xy - 2\lambda z \qquad \Rightarrow \qquad xy - 2\lambda z = 0 \qquad \qquad \text{...(3)}$

$F_\lambda = -x - 9y^2 - z^2 + 4 \qquad \Rightarrow \qquad x^2 + 9y^2 + z^2 - 4 = 0 \qquad \text{...(4)}$

Now multiplying equation (1) by x, equation (2) by y and equation (3) by z.

We get

$$xyz = \lambda x \qquad \qquad ...(5)$$

$$xyz = 18\lambda y^2 \qquad \qquad ...(6)$$

$$xyz = 2\lambda z^2 \qquad \qquad ...(7)$$

by equation (5) and (6)

$$x\lambda = 18\lambda y^2 \quad \Rightarrow \quad (x - 18y^2)\lambda = 0$$

$$\Rightarrow \quad x = 18y^2 \text{ or } \lambda = 0$$

by equation (6) and (7)

$$18y^2\lambda = 2z^2\lambda \quad \Rightarrow \quad (18y^2 - 2z^2)\lambda = 0$$

$$\Rightarrow \quad z^2 = 9y^2 \text{ or } \lambda = 0$$

Now we have two possibilities

Either $\lambda = 0$ or we have $x = 18y^2$ and $z^2 = 9y^2$.

Let $\lambda = 0$, then by equation (1), (2) and (3) we have

$$yz = 0 \quad \Rightarrow \quad y = 0 \text{ or } z = 0 \qquad \qquad ...(8)$$

$$xz = 0 \quad \Rightarrow \quad x = 0 \text{ or } z = 0 \qquad \qquad ...(9)$$

$$xy = 0 \quad \Rightarrow \quad x = 0 \text{ or } y = 0 \qquad \qquad ...(10)$$

from this we noticed that we can't have all three of the variables be zero but we could have two of then be zero. So this leads to the following two cases that we can plug into the constraint to find the value of the third variable.

$$y = 0, x = 0 \quad \Rightarrow \quad z^2 = 4 \quad \Rightarrow \quad z = \pm 2$$

$$y = 0, z = 0 : x = 4$$

So this gives us the following three absolute extrema

$$(0, 0, -2), (0, 0, 2) \text{ and } (4, 0, 0)$$

Now, let's check the second possibility from equation (8) z = 0. In this case the (9) equation while be 0 = 0 and so will not be any use. Then (10) however, has the possibilities of x = 0 or y = 0.

$$z = 0, x = 0 : 9y^2 = 4 \quad \Rightarrow \quad y = \pm \frac{2}{3}$$

This leads to two more absolute extrema

$$\left(0, -\frac{2}{3}, 0\right) \text{ and } \left(0, \frac{2}{3}, 0\right)$$

we had another possibility $x = 18y^2$ and $z^2 = 9y^2$

In this case we can plug each of these directly into the constraint to get the following

$$18y^2 + 9y^2 + 9y^2 = 36y^2 = 4 \quad \Rightarrow \quad y = \pm \frac{1}{3}$$

Now $\quad x = 18\left(\frac{1}{9}\right) = 2 \quad z^2 = 9\left(\frac{1}{9}\right) = 1 \quad \Rightarrow \quad z = \pm 1$

Therefore we get the following four absolute extrema

$$\left(2, -\frac{1}{3}, -1\right), \left(2, -\frac{1}{3}, 1\right), \left(2, \frac{1}{3}, -1\right) \text{ and } \left(2, \frac{1}{3}, 1\right)$$

Now $\quad f(0, 0, \pm 2) = f\left(0, \pm\frac{2}{3}, 0\right) = f(4, 0, 0) = 0$

$$f\left(2, -\frac{1}{3}, 1\right) = f\left(2, \frac{1}{3}, -1\right) = -\frac{2}{3}$$

$$f\left(2, -\frac{1}{3}, -1\right) = f\left(2, \frac{1}{3}, 1\right) = \frac{2}{3}$$

The absolute minimum is $-\frac{2}{3}$ which occurs at $\left(2, -\frac{1}{3}, 1\right)$ and $\left(2, \frac{1}{3}, -1\right)$.

1. ANTIDERIVATIVES - DIFFERENTIATION IN REVERSE

Consider the function $F(x) = 3x^2 + 7x - 2$. Suppose we write its derivative as $f(x)$, i.e. $f(x) = \dfrac{dF}{dx}$. We already know how to find this derivative by differentiating term by term to obtain $f(x) = \dfrac{dF}{dx} = 6x + 7$. This process is illustrated in Fig..

<div align="center">

differentiate

$F(x) = 3x^2 + 7x - 2 \qquad f(x) = 6x + 7$

antidifferentiate

</div>
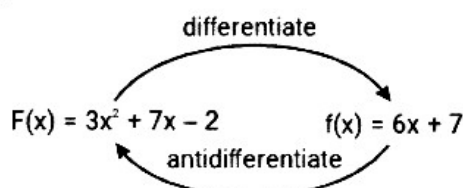
Fig. : The function F(x) is an antiderivative of f(x)

Suppose now that we work back to front and ask ourselves which function or functions could possibly have $6x + 7$ as a derivative. Clearly, one answer to this question is the function $3x^2 + 7x - 2$. We say that $F(x) = 3x^2 + 7x - 2$ is an **antiderivative** of $f(x) = 6x + 7$.

There are however other functions which have derivative $6x + 7$. Some of these are

$$3x^2 + 7x + 3, \qquad 3x^2 + 7x, \qquad 3x^2 + 7x - 11$$

The reason why all of these functions have the same derivative is that the constant term disappears during differentiation. So, all of these are antiderivatives of $6x + 7$. Given any antiderivative of $f(x)$, all others can be obtained by simply adding a different constant. In order words, if $F(x)$ is an antiderivative of $f(x)$, then so too is $F(x) + C$ for any constant C.

Geometrical Interpretation of Integration

Let $f(x)$ be a given continuous function and $F(x)$ one of its antiderivatives such that

$$\int f(x)\, dx = F(x) + c$$

If $\quad y = \int f(x)\, dx = F(x) + c \qquad\qquad$ ...(1)

then $\quad y = F(x) + c$ represents a family of "parallel" curves.

It is clear from (1), we have

Here $\quad c_4 > c_3 > c_2 > c_1 > 0$.
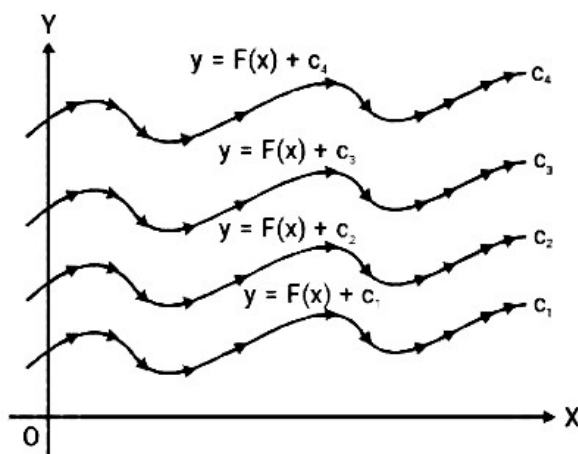
Fig.

Example : Find the integral curve of the equation $dy = dx$.

Solution : Given $dy = dx$ ...(1)

Comparing (1) with $dy = f(x) dx$

$\therefore$ $f(x) = 1$
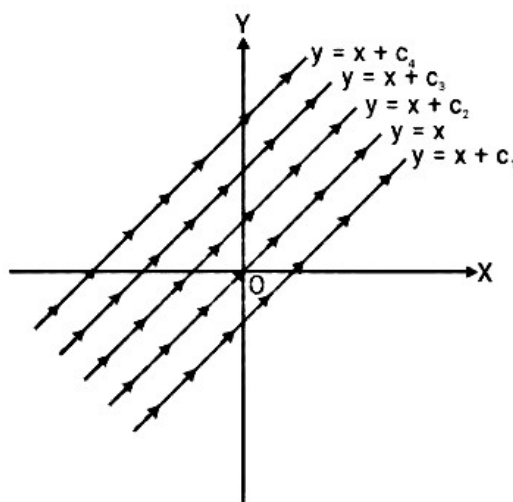

Fig.

or $y = \int 1 . dr = x + c$ $\left(\because \dfrac{d}{dx}(x) = 1 \right)$

the integral curves are parallel lines with slope of all arrows is unity.

Example : Find the integral curve of the equation $dy = 2x\, dx$.
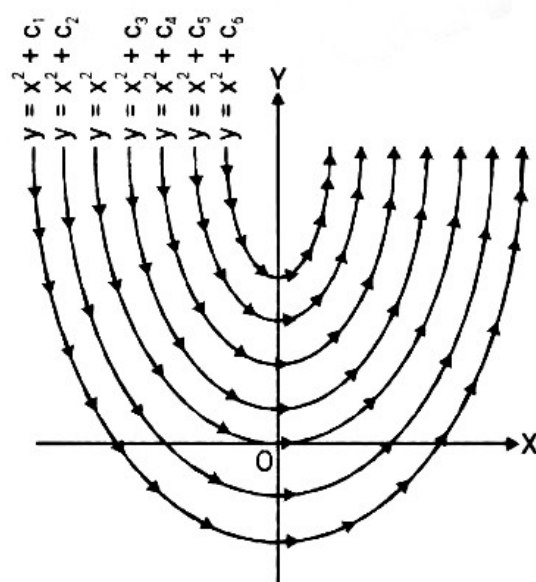
Solution. Given $dy = 2x\, dx$ ...(1)

Fig.

Comparing (1) with $dy = f(x) dx$

$\therefore$ $f(x) = 2x$

or $y = \int 2x\, dx = x^2 + c$ $\left(\because \dfrac{d}{dx}(x^2) = 2x \right)$

represents a family of "parallel" curves.

Example : **(a)** Differentiate $F(x) = 4x^3 - 7x^2 + 12x - 4$ to find $f(x) = \dfrac{dF}{dx}$.

(b) Write down several antiderivatives of $f(x) = 12x^2 - 14x + 12$.

Solution. **(a)** Differentiating $F(x) = 4x^3 - 7x^2 + 12x - 4$ we find $f(x) = \dfrac{dF}{dx} = 12x^2 - 14x + 12$. We can deduce from this that an antiderivative of $12x^2 - 14x + 12$ is $4x^3 - 7x^2 + 12x - 4$.

(b) All other antiderivatives of $f(x)$ will take the form $F(x) + C$ where C is a constant. So, the following are all antiderivatives of $f(x)$:

$$4x^3 - 7x^2 + 12x - 4, \quad 4x^3 - 7x^2 + 12x - 10, \quad 4x^3 - 7x^2 + 12x, \quad 4x^3 - 7x^2 + 12x + 3$$

From these examples we deduce the following important observation.

Key Point

A function $F(x)$ is an antiderivative of $f(x)$ if $\dfrac{dF}{dx} = f(x)$.

If $F(x)$ is an antiderivative of $f(x)$ then so too is $F(x) + C$ for any constant C.

Derivatives	**Integrals (Anti derivatives)**
(i) $\dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1}\right) = x^n$;	$\displaystyle\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C, n \neq -1$
Particularly, we note that	
$\dfrac{d}{dx}(x) = 1$;	$\displaystyle\int dx = x + C$
(ii) $\dfrac{d}{dx}(\sin x) = \cos x$;	$\displaystyle\int \cos x \, dx = \sin x + C$
(iii) $\dfrac{d}{dx}(-\cos x) = \sin x$;	$\displaystyle\int \sin x \, dx = -\cos x + C$
(iv) $\dfrac{d}{dx}(\tan x) = \sec^2 x$;	$\displaystyle\int \sec^2 x \, dx = \tan x + C$
(v) $\dfrac{d}{dx}(-\cot x) = \csc^2 x$;	$\displaystyle\int \csc^2 x \, dx = -\cot x + C$
(vi) $\dfrac{d}{dx}(\sec x) = \sec x \tan x$;	$\displaystyle\int \sec x \tan x \, dx = \sec x + C$
(vii) $\dfrac{d}{dx}(-\csc x) = \csc x \cot x$;	$\displaystyle\int \csc x \cot x \, dx = -\csc x + C$
(viii) $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$;	$\displaystyle\int \dfrac{d}{\sqrt{1-x^2}} = \sin^{-1} x + C$
(ix) $\dfrac{d}{dx}(-\cos^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$;	$\displaystyle\int \dfrac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$
(x) $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$;	$\displaystyle\int \dfrac{dx}{1+x^2} = \tan^{-1} x + C$

(xi) $\dfrac{d}{dx}\left(-\cot^{-1} x\right) = \dfrac{1}{1+x^2}$; $\displaystyle\int \dfrac{dx}{1+x^2} = -\cot^{-1} x + C$

(xii) $\dfrac{d}{dx}\left(\sec^{-1} x\right) = \dfrac{1}{x\sqrt{x^2-1}}$; $\displaystyle\int \dfrac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$

(xiii) $\dfrac{d}{dx}\left(-\cosec^{-1} x\right) = \dfrac{1}{x\sqrt{x^2-1}}$; $\displaystyle\int \dfrac{dx}{x\sqrt{x^2-1}} = -\cosec^{-1} x + C$

(xiv) $\dfrac{d}{dx}\left(e^x\right) = e^x$; $\displaystyle\int e^x dx = e^x + C$

(xv) $\dfrac{d}{dx}\left(\log|x|\right) = \dfrac{1}{x}$; $\displaystyle\int \dfrac{1}{x} dx = \log|x| + C$

(xvi) $\dfrac{d}{dx}\left(\dfrac{a^x}{\log a}\right) = a^x$; $\displaystyle\int a^x dx = \dfrac{a^x}{\log a} + C$

Examples : Let I = $\displaystyle\int\left(\dfrac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x}\right) dx$

$= \displaystyle\int \dfrac{(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x(\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x} dx$

$= \displaystyle\int \dfrac{1 - 3\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} dx$

$= \displaystyle\int \dfrac{\sin^2 x + \cos^2 x - 3\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} dx$

$= \displaystyle\int \dfrac{1}{\cos^2 x} dx + \int \dfrac{1}{\sin^2 x} dx - \int 3\, dx$

$= \displaystyle\int \sec^2 x\, dx + \int \cosec^2 x\, dx - 3\int dx$

$=$ tan x − cot x − 3x + c

Examples : Let I = $\displaystyle\int\left(\dfrac{\cos 2x - \cos 2\alpha}{\sin x - \sin\alpha}\right) dx$

$= \displaystyle\int \dfrac{-2(\sin^2 x - \sin^2 \alpha)}{(\sin x - \sin\alpha)} dx$

$= \displaystyle\int \dfrac{-2(\sin^2 x - \sin^2 \alpha)}{(\sin x - \sin\alpha)} dx$

$= -2\displaystyle\int \dfrac{(\sin x + \sin\alpha)(\sin x - \sin\alpha)}{(\sin x - \sin\alpha)} dx$

$= -2\displaystyle\int (\sin x + \sin\alpha) dx$

$= -2\displaystyle\int \sin x\, dx - 2\sin\alpha \int dx$

$=$ 2 cos x − 2x sin α + c

Examples : Evaluate :

$$\int \left\{ \frac{5\cos^3 x + 2\sin^3 x}{2\sin^2 x \cos^2 x} + \sqrt{(1 + \sin 2x)} + \left(\frac{1 + 2\sin x}{\cos^2 x} \right) + \frac{1 - \cos 2x}{1 + \cos 2x} \right\} dx$$

Solution. Let $I = \int \left\{ \frac{5\cos^3 x + 2\sin^3 x}{2\sin^2 x \cos^2 x} + \sqrt{(1 + \sin 2x)} + \left(\frac{1 + 2\sin x}{\cos^2 x} \right) + \frac{1 - \cos 2x}{1 + \cos 2x} \right\} dx$

$$= \int \left\{ \frac{5}{2} \cos ec\, x \cot x + \sec x \tan x + (\cos x + \sin x) + (\sec^2 x + 2\sec x \tan x) + \tan^2 x \right\} dx$$

$$= \int \left\{ \frac{5}{2} \cos ec\, x \cot x + 3\sec x \tan x + \cos x + \sin x + 2\sec^2 x - 1 \right\} dx$$

$$= \frac{5}{2} \int \cos ec\, x \cot x\, dx + 3 \int \sec x \tan x\, dx + \int \cos x\, dx + \int \sin x\, dx + 2 \int \sec^2 x\, dx - \int 1.dx$$

$$= -\frac{5}{2} \cos ec + 3 \sec x + \sin x - \cos x + 2 \tan x - x + c$$

Properties of the Indefinite Integral

The differential of an indefinite integral is equal to the element of integration, and the derivative of an indefinite integral is equal to the integral.

Thus, we have $d \int f(x)\, dx = f(x)\, dx$

or $\quad \frac{d}{dx} \int f(x)\, dx = f(x) \quad$ or $\quad \left\{ \int f(x) dx \right\} = f(x)$

e.g., $\quad d \int 3x^2 dx = d\,(x^3 + c) + 3x^2\, dx$

or $\quad \frac{d}{dx} \int 3x^2\, dx = 3x^2$

The indefinite integral of the differential of a continuously differentiable function is equal to this function, but introduces an arbitrary additive constant.

Thus, we have $\int df(x) = f(x) + c$

e.g., $\quad \int d \cos x = \cos x + c$

A non-zero constant factor may be taken outside the sign of the integral,
i.e., if constant $a \neq 0$, then

$$\int a f(x)\, dx = a \int f(x)\, dx$$

e.g., $\quad \int 2x\, dx = 2 \int x\, dx = 2 \left(\frac{x^2}{2} + c \right) = x^2 + 2c = x^2 + c_1$

where $c_1 + 2c$

The integral of an algebraic sum is equal to the sum of the integrals of the summands for n summands:

$$\int \{f_1(x) \pm f_2(x) \pm \ldots \pm f_n(x)\}dx = \int f_1(x)\,dx \pm \int f_2(x)\,dx \pm \ldots \pm \int f_n(x)\,dx + c$$

e.g., $\int (6x^2 - 4x + 5)dx = \int 6x^2 dx - \int 4x\,dx + \int 5\,dx$

$$= 6\int x^2 dx - 4\int x\,dx + 5\int dx$$

$$= \left(6 \cdot \frac{x^3}{3} + c_1\right) - \left(4 \cdot \frac{x^2}{2} + c_2\right) + (5x + c_3)$$

$$= 2x^3 - 2x^2 + 5x + c$$

where $c = c_1 - c_2 + c_3$

Note : The constant term for every integral is adjoined in the last after all integrations have been performed.

Methods of Integration

To find the integral of complex problems (The integral is not a derivative of a known function). Following methods are used.

(I) Integration by Substitution or by change of the independent variable :

If the independent variable x in $\int f(x)\,dx$ be changed to a new variable t, then we substitute $x = \phi(t)$, where $\phi(t)$ is a continuous differentiable function, then

$$dx = d(\phi(t))$$
$$= \phi'(t)\,dt$$

then we have

$$\int f(x)\,dx = \int f\{\phi(t)\}\,\phi'(t)\,dt$$

which is either a standard form or is easier to integrate. Here after integration we revert back to the old variable x by the inverse substitution $t = \phi^{-1}(x)$.

(a) Three fundamental deductions of the method of substitution :

Deduction I : $\int \{f(x)\}^n f'(x)dx = \dfrac{\{f(x)\}^{n+1}}{(n+1)} + c$ $(n \neq 1)$

(Power formula)

Deduction II : $\int \dfrac{f'(x)}{f(x)}dx = \ln |f(x)| + c$

Deduction III : $\int \dfrac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c$

(b) Standard Substitutions :

	Expression	Substitution
1.	$a^2 - x^2$ or $\sqrt{(a^2 - x^2)}$	$x = a \sin\theta$ or $a \cos\theta$
2.	$a^2 + x^2$ or $\sqrt{(a^2 + x^2)}$	$x = a \tan\theta$ or $a \cot\theta$
3.	$x^2 - a^2$ or $\sqrt{(x^2 - a^2)}$	$x = a \sec\theta$ or $a \csc\theta$
4.	$\sqrt{(a+x)}$ or $\sqrt{(a-x)}$ or $\sqrt{\left(\dfrac{a+x}{a-x}\right)}$ or $\sqrt{\left(\dfrac{a-x}{a+x}\right)}$	$x = a \cos\theta$
5.	$\sqrt{\left(\dfrac{x-a}{b-x}\right)}$ or $\sqrt{(x-a)(b-x)}$	$x = a \cos^2\theta + b \sin^2\theta$
6.	$\sqrt{\left(\dfrac{x-a}{x-b}\right)}$ or $\sqrt{(x-a)(x-b)}$	$x = a \sec^2\theta - b \tan^2\theta$
7.	$\sqrt{(2ax - x^2)}$	$x = a (1 - \cos\theta)$
8.	$\dfrac{1}{\sqrt{(x-a)(x-b)}}$	$x - a = t^2$ or $x - b = t^2$
9.	$\dfrac{1}{(x+a)^m (x+b)^n}$, $m, n \in N$ (and > 1)	$(x + a) = t (x + b)$
10.	$(x+a)^{-(1/n)-1}(x+b)^{(1/n)-1}$ or $\left(\dfrac{x+b}{x+a}\right)^{(1/n)-1} \dfrac{1}{(x+a)^2}$ $(n \in N$ and $> 1)$	$\dfrac{x+b}{x+a} = t$

(c) Extended forms of fundamental formulae :

If $\quad \int f(x)\,dx = F(x)$ then we find $\int f(ax+b)\,dx$.

Let $\quad I = \int f(ax+b)\,dx$ ...(1)

Putting $ax + b = t$ so that $a\,dx = dt \quad \Rightarrow \quad dx = \dfrac{1}{a}\,dt$

then $\quad I = \int f(t)\cdot\dfrac{1}{a}\,dt$

$= \dfrac{1}{a}\int f(t)\,dt$

$= \dfrac{1}{a}F(t)$ $[\because \int f(t)\,dt = F(t)]$

$= \dfrac{1}{a}F(ax+b)$ ...(2)

From (1) and (2) we get $\int f(ax+b)\,dx = \dfrac{1}{a}F(ax+b)$

Thus to evaluate $\int f(ax+b)\,dx$ supposing in mind $ax+b$ as a variable like x and divide it by the coefficient of x in $ax+b$ i.e., a.

From this we obtain the following results :

(i) $\quad \int (ax+b)^n\,dx = \dfrac{(ax+b)^{(n-1)}}{a(n+1)} + c,\ n \neq -1$

(ii) $\quad \int \dfrac{dx}{(ax+b)^n} = \dfrac{-1}{a(n-1)(ax+b)^{n-1}} + c,\ n \neq 1$

(iii) $\quad \int \dfrac{dx}{ax+b} = \dfrac{1}{a} \ln |ax+b| + c$

(iv) $\quad \int e^{(ax+b)}\,dx = \dfrac{1}{a} e^{(ax+b)} + c$

	Integral	Method of integration
(i)	$\int f\left(x + \dfrac{1}{x}\right)\cdot\left(1 - \dfrac{1}{x^2}\right)dx$	Put $x + \dfrac{1}{x} = t \Rightarrow \left(1 - \dfrac{1}{x^2}\right)dx = dt$
(ii)	$\int f\left(x - \dfrac{1}{x}\right)\cdot\left(1 + \dfrac{1}{x^2}\right)dx$	Put $x - \dfrac{1}{x} = t \Rightarrow \left(1 + \dfrac{1}{x^2}\right)dx = dt$
(iii)	$\int f\left(x^2 + \dfrac{1}{x^2}\right)\cdot\left(x - \dfrac{1}{x^3}\right)dx$	Put $x^2 + \dfrac{1}{x^2} = t \Rightarrow \left(x - \dfrac{1}{x^3}\right)dx = \dfrac{1}{2}dt$
(iv)	$\int f\left(x^2 - \dfrac{1}{x^2}\right)\cdot\left(x + \dfrac{1}{x^3}\right)dx$	Put $x^2 - \dfrac{1}{x^2} = t \Rightarrow \left(x + \dfrac{1}{x^3}\right)dx = \dfrac{1}{2}dt$

Examples : Evaluate : $\displaystyle\int \dfrac{(x^2-1)\,dx}{(x^4+3x^2+1)\tan^{-1}\left(x+\dfrac{1}{x}\right)}$

Solution. Let $I = \displaystyle\int \dfrac{(x^2-1)\,dx}{(x^4+3x^2+1)\tan^{-1}\left(x+\dfrac{1}{x}\right)}$

$= \displaystyle\int \dfrac{\left(1 - \dfrac{1}{x^2}\right)dx}{\left(x^2+3+\dfrac{1}{x^2}\right)\tan^{-1}\left(x+\dfrac{1}{x}\right)}$

$= \displaystyle\int \dfrac{\left(1 - \dfrac{1}{x^2}\right)dx}{\left\{\left(x+\dfrac{1}{x}\right)^2 + 1\right\}\tan^{-1}\left(x+\dfrac{1}{x}\right)}$

Put $x + \dfrac{1}{x} = t \Rightarrow \left(1 - \dfrac{1}{x^2}\right)dx = dt$ then $I = \displaystyle\int \dfrac{dt}{(t^2+1)\tan^{-1}t}$

Again put $\tan^{-1} t = z$

$\Rightarrow \quad \dfrac{dt}{(1+t^2)} = dz$ then we get

$I = \displaystyle\int \dfrac{dz}{z} = \ln |z| + c \ ln \ | \tan^{-1} t | + c$ $\qquad\qquad [\because \ z = \tan^{-1} t]$

$= ln \left| \tan^{-1}\left(x + \dfrac{1}{x}\right) \right| + c$

Examples : Evaluate : $\displaystyle\int \dfrac{\sqrt{x}}{\sqrt{(a^3 - x^3)}} dx$.

Solution. Let $I = \displaystyle\int \dfrac{\sqrt{x}}{\sqrt{(a^3 - x^3)}} dx$

Since integral of $\sqrt{x}$ is $\dfrac{x^{3/2}}{3/2}$

then put $x^{3/2} = t \Rightarrow x^3 = t^2$

or $\qquad \dfrac{3}{2}\sqrt{x} \ dx = dt \quad \Rightarrow \quad \sqrt{x} \ dx = \dfrac{2}{3} dt$

then we get $I = \dfrac{3}{2}\displaystyle\int \dfrac{dt}{\sqrt{a^3 - t^2}}$

$= \dfrac{2}{3}\displaystyle\int \dfrac{dt}{\sqrt{(a^{3/2})^2 - t^2}}$

$= \dfrac{2}{3}\sin^{-1}\left(\dfrac{t}{a^{3/2}}\right) + c$

$= \dfrac{2}{3}\sin^{-1}\left(\dfrac{x^{3/2}}{a^{3/2}}\right) + c$

$= \dfrac{2}{3}\sin^{-1}\left(\dfrac{x}{a}\right)^{3/2} + c$

II. Integration by Parts

Let u and v be two functions of x, then we know that from differential calculus

$\dfrac{d}{dx}(u \cdot v) = u \cdot \dfrac{dv}{dx} + v \cdot \dfrac{du}{dx}$

Integrating both sides w. r. t. x, we get

$uv = \displaystyle\int u\dfrac{dv}{dx} dx + \int v\dfrac{du}{dx} dx$

or $\qquad \displaystyle\int u \cdot \dfrac{dv}{dx} dx = uv - \int v\dfrac{du}{dx} dx$ $\qquad\qquad\qquad$ ...(1)

Now put u = f(x) and v = $\int g(x)\, dx$,

so that $\dfrac{dv}{dx} = g(x)$

Substituting these value in (1), we get

$$\int f(x).g(x)dx = f(x) . \left\{\int g(x)\, dx\right\} - \int\left[f'(x)\left\{\int g(x)\, dx\right\}\right]dx$$

The above formula can be put in words i.e.,

The integral of the product of two functions = (first function) × (integral of the second function) – Integral of {diff. coeff. of the first function × Integral of the second function}

(a) How to Choose 1st and 2nd function

(i)   If the two functions are of different types take that function as 1st which comes first in the word **ILATE** where **I** stands for inverse Trigonometric function, **L** stands for logarithmic function, **A** stands for algebraic function, **T** stands for trigonometric function and **E** stands for exponential function.

(ii)   If both functions are algebraic take that function as 1st whose differential coefficient is simpler, and take remaining as the 2nd function.

(iii) If both function are trigonometrical take that function as 2nd whose integral is simpler and take remaining as the 1st function.

(iv)   If integral contains only one function which can not be directly integral.

(i.e., $\ln |x|$, $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, ... etc.)

then second function be chosen as unity.

Note : The formula of integration by parts can be applied more than once if necessary.

Examples : Evaluate : $\int x^n \ln x\, dx$, $n \neq -1$ and $x > 0$

Solution. Let $I = \int x^n \ln x\, dx$

Here x^n is algebraic and $\ln x$ is logarithmic function, in **ILATE** rule L come before A, therefore Integrating by parts taking x^n as second function, we have

$$I = \ln x \cdot \frac{x^{n+1}}{(n+1)} - \int \frac{1}{x} \cdot \frac{x^{n+1}}{(n+1)} dx$$

$$= \frac{1}{(n+1)} \cdot x^{n+1} \cdot \ln x - \frac{1}{(n+1)} \int x^n\, dx$$

$$= \frac{x^{n+1} \ln x}{(n+1)} - \frac{x^{n+1}}{(n+1)^2} + c$$

$$= \frac{x^{n+1}}{(n+1)} \left\{ \ln x - \frac{1}{(n+1)} \right\} + c$$

Examples : Evaluate : $\int \sin^{-1} x\, dx$.

Solution. Let $I = \int (\sin^{-1} x)dx$

Here there is only one function which can not be directly integrated then unity should be taken as the 2nd function we have

$I = \int \sin^{-1} x . 1 . dx$

$= \sin^{-1} x . x - \int \dfrac{1 . x}{\sqrt{(1-x^2)}} dx$

$= x \sin^{-1} x - \int \dfrac{x}{\sqrt{(1-x^2)}} dx$

In integral put $1 - x^2 = t$ $\Rightarrow$ $x\ dx = -\dfrac{1}{2} dt$

then $I = x \sin^{-1} x + \dfrac{1}{2} \int \dfrac{dt}{\sqrt{t}}$

$= x \sin^{-1} x + \sqrt{t} + c$

$= x \sin^{-1} x + \sqrt{(1-x^2)} + c$ $(\because\ t = 1 - x^2)$

Note :

(i) $\int \dfrac{x}{\sqrt{(a^2 - x^2)}} dx = -\sqrt{(a^2 - x^2)}$

(ii) $\int \dfrac{x}{\sqrt{a^2 + x^2}} dx = \sqrt{(a^2 + x^2)}$ (Remember)

(iii) $\int \dfrac{x}{\sqrt{(x^2 - a^2)}} dx = \sqrt{(x^2 - a^2)}$

(b) Cancellation of Integrals

Sometimes we split the integrand into the sum of two parts such that the integration of one of them by parts cancels the other.

$$\int e^x (f(x) + f'(x))\ dx = e^x\ f(x) + c$$

Proof : $\int e^x (f(x) + f'(x)) dx = \int e^x f(x) dx + \int e^x f'(x) dx$

Integrating 1st term by parts taking e^x as second function we have

$= f(x)\ e^x - \int f'(x) . e^x\ dx + \int e^x f'(x) + c$

$= e^x\ f(x) + c$ (The last two integrals cancel each other).

Alternative Proof :

$\therefore \quad \dfrac{d}{dx} \{e^x\ f(x)\} = e^x\ f(x) + e^x\ f'(x) = e^x\ \{f(x) + f'(x)\}$

On integrating both sides, we have

$\int e^x (f(x) + f'(x)) dx = e^x f(x) + c$

Note :

$\int e^{\lambda x} \{f(\lambda x) + f'(\lambda x)\} dx + e^{\lambda x} f(\lambda x) + c$

This is every important and the students can use this as a formula.

Examples : Evaluate : $\int \left\{ \dfrac{1}{\ln x} - \dfrac{1}{(\ln x)^2} \right\} dx$.

Solution. Let $I = \int \left\{ \dfrac{1}{\ln x} - \dfrac{1}{(\ln x)^2} \right\} dx$

Put $\ln x = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$

then $\quad I = \int \left(\dfrac{1}{t} - \dfrac{1}{t^2} \right) e^t \, dt$

$= \int e^t \left\{ \dfrac{1}{t} + \left(-\dfrac{1}{t^2} \right) \right\} dt$

$= \int e^t \{ f(t) + f'(t) \} dt = e^t f(t) + c$

where $f(t) = \dfrac{1}{t}$ and $f'(t) = -\dfrac{1}{t^2}$ $\qquad \therefore \qquad I = e^t \cdot \dfrac{1}{t} + c = \dfrac{x}{\ln x} + c$

Note : If logarithmic function or inverse circular function presents in the denominator of integrand

i.e., $\quad \int \dfrac{f(t)}{\ln x} dx \qquad\qquad$ or $\qquad\qquad \int \dfrac{f(x)}{\sin^{-1} x} dx$

then put it equal to t.

(d) Integrals of $e^{ax} \cos bx$ and $e^{ax} \sin bx$:

Prove : $\int e^{ax} \cos bx \, dx = \dfrac{e^{ax}(a \cos bx + b \sin bx)}{(a^2 + b^2)} + c$

and $\quad \int e^{ax} \sin bx \, dx = \dfrac{e^{ax}(a \sin bx - b \cos bx)}{(a^2 + b^2)} + c$

Proof : Let $I = \int e^{ax} \cos bx \, dx \qquad\qquad\qquad\qquad$ ...(1)

Integrating by parts taking e^{ax} as 2nd function, we have

$I = \cos bx \left(\dfrac{e^{ax}}{a} \right) - \int (-b \sin bx) \left(\dfrac{e^{ax}}{a} \right) dx$

$= \dfrac{1}{a} e^{ax} \cos bx + \dfrac{b}{a} \int \sin bx \, e^{ax} \, dx \qquad\qquad\qquad$ ...(2)

Again integrating by parts taking e^{ax} as 2nd function, then

$I = \dfrac{1}{a} e^{ax} \cos bx + \dfrac{b}{a} \left\{ \sin bx \left(\dfrac{e^{ax}}{a} \right) - \int b \cos bx \cdot \dfrac{e^{ax}}{a} dx \right\}$

$= \dfrac{1}{a} e^{ax} \cos bx + \dfrac{b}{a^2} \sin bx \, e^{ax} - \dfrac{b^2}{a^2} I \qquad\qquad$ (from (1))

or $\quad I\left(1+\dfrac{b^2}{a^2}\right)=\dfrac{e^{ax}}{a^2}(a\cos bx + b\sin bx)$

$\Rightarrow\quad I=\dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

Examples : Evaluate : $\int x^2\ e^{2x\cos\alpha}\sin(2x\sin\alpha)\,dx$

Solution. Let $I=\int x^2\ e^{2x\cos\alpha}\sin(2x\sin\alpha)\,dx$

$$I=\int x^2\ e^{ax}\sin(bx)\,dx$$

where $a=2\cos\alpha$ and $b=2\sin\alpha$

$\therefore\quad \sqrt{(a^2+b^2)}=\sqrt{(2\cos\alpha)^2+(2\sin\alpha)^2}=\sqrt{4\cos^2\alpha+4\sin^2\alpha}=\sqrt{4(\cos^2\alpha+\sin^2\alpha)}=2$

and $\tan^{-1}\left(\dfrac{b}{a}\right)=\tan^{-1}\left(\dfrac{2\sin\alpha}{2\cos\alpha}\right)=\tan^{-1}(\tan\alpha)=\alpha$ $\qquad$ ...(1)

Integrating by parts

$$I=\int x^2 e^{ax}\sin bx\,dx = x^2\int e^{ax}\sin bx\,dx - \int\left(2x\int e^{ax}\sin bx\,dx\right)dx$$

We know that

$$\int e^{ax}\sin bx = \dfrac{e^{ax}}{a^2+b^2}(a\sin bx + b\cos bx)$$

$$\int e^{ax}\cos bx = \dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$I = x^2\dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) - \int 2x\dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\,dx$$

$$I = \dfrac{x^2 e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) - \dfrac{2}{a^2+b^2}\int x e^{ax}(a\sin bx - b\cos bx)\,dx$$

Let $\quad I_1 = \int x e^{ax}(a\sin bx - b\cos bx)\,dx$

$= \int x\ e^{ax}\ a\sin bx\,dx - \int x\ e^{ax}\ b\cos bx\,dx$

$= ax\int e^{ax}\ \sin bx\,dx - \int\dfrac{ae^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\,dx$

$\qquad -bx\int e^{ax}\cos bx\,dx + \int b\,\dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\,dx$

$= ax\,\dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) - \dfrac{a}{a^2+b^2}\int e^{ax}(a\sin bx - b\cos bx)\,dx$

$\qquad -bx\,\dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin dx) + \dfrac{b}{a^2+b^2}\int e^{ax}(a\cos bx + b\sin bx)\,dx$

$$= \frac{(a-b)x}{a^2+b^2} e^{ax}\left((a+b)\sin bx + (a-b)\cos bx\right)$$

$$+ \frac{(b-a)}{a^2+b^2} \int \left(e^{ax}\left((a+b)\sin bx + (a-b)\cos bx\right)\right) dx$$

Now,   Let $I_2 = \int e^{ax}\left((a+b)\sin bx + (a-b)\cos bx\right)dx$

$I_2 = (a+b) \int e^{ax}\sin bx + dx + (a-b)\int e^{ax}\cos bx\, dx$

$I_2 = (a+b)\ \dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + (a-b)\dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$I = \dfrac{x^2 e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) - \dfrac{2}{\left(a^2+b^2\right)}\left[\dfrac{(a-b)x}{a^2+b^2} e^{ax}\left((a+b)\sin bx\right)\right.$

$$\left.(a-b)\cos bx + \frac{(b-a)}{a^2+b^2}\left(\frac{e^{ax}}{a^2+b^2}\, 2a\left((a+b)\sin bx + (a-b)\cos bx\right)\right)\right]$$

Now substituting $a = 2\cos\alpha$ and $b = 2\sin\alpha$ in above integral we get the desired result.

(III) Integration of Rational Fractions

An expression of the $f(x)/g(x)$, where $f(x)$ and $g(x)$ are polynomials in x, is called a rational fraction.

i.e., $\dfrac{f(x)}{g(x)} = \dfrac{a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n}{b_0 x^m + b_1 x^{m-1} + \ldots + b_{m-1}x + b_m},$

where $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m$ are constants and $m, n \in N$.

Type of fractions

(i) Proper fraction :

If degree of $f(x) <$ degree of $g(x)$, $f(x)/g(x)$ is called a proper fraction.

Example : $\dfrac{x+2}{x^2+3x+5}$ is a proper fraction.

∵   Here degree of numerator = 1 and degree of denominator = 2

∴   degree of numerator < degree of denominator.

(ii) Improper fraction :

If degree of $f(x) \geq$ degree of $g(x)$, $f(x)/g(x)$ is called an improper fraction.

then divide $f(x)$ by $g(x)$ so that $\dfrac{f(x)}{g(x)} = h(x) + \dfrac{\phi(x)}{g(x)}$

where $h(x)$ is an integral function and $\dfrac{\phi(x)}{g(x)}$ is a proper fraction.

Example : $\dfrac{x^2+1}{x+1}$ is an improper fraction.

∵   Here degree of numerator = 2 and degree of denominator = 1

∴   degree of numerator > degree of denominator

Then divide $(x^2 + 1)$ by $(x + 1)$ such that

$$x + 1 \overline{\smash{\big)}\ x^2 + 1\ } \quad x - 1$$
$$\underline{\begin{array}{r} x^2 + x \\ -\ \ - \end{array}}$$
$$\begin{array}{r} -x + 1 \\ -x - 1 \\ +\ \ + \end{array}$$
$$\underline{}$$
$$2$$

$\therefore \qquad \dfrac{x^2 + 1}{x + 1} = x - 1 + \dfrac{2}{x + 1}$

Here $(x - 1)$ is an integral function and $\dfrac{2}{x + 1}$ is a proper fraction.

Any proper fraction can be expressed as the sum of two or more simple fractions. Each such fraction is called a partial fraction and the process of obtaining them is called the resolution or decomposition of $f(x)/g(x)$ into partial fractions.

Case I : Integration of fractions with non-repeated linear factors in the denominator.

An non-repeated linear factor $(x - a)$ of denominator then corresponds a partial fraction of the form $\dfrac{A}{(x - a)}$ where A is constant to be determined.

If $g(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$ then we assume that

$$\frac{f(x)}{g(x)} \equiv \frac{A_1}{(x - a_1)} + \frac{A_2}{(x - a_2)} + \ldots + \frac{A_n}{(x - a_n)}$$

where $A_1, A_2, \ldots, A_n$ are constants, can be determined equating the numerator of L.H.S. to the numerator of R.H.S. (after L.C.M.) and substituting $x = a_1, a_2, \ldots, a_n$.

Examples : Evaluate : $\displaystyle\int \frac{x^2}{(x - 1)(x - 2)(x - 3)} dx$.

Solution. Let $I = \displaystyle\int \frac{x^2}{(x - 1)(x - 2)(x - 3)} dx$

Here $\dfrac{x^2}{(x - 1)(x - 2)(x - 3)} \equiv \dfrac{A}{(x - 1)} + \dfrac{B}{(x - 2)} + \dfrac{C}{(x - 3)}$ (say)

then $\quad x^2 = A(x - 2)(x - 3) + B(x - 3)(x - 1) + C(x - 1)(x - 2)$ ...(1)

Putting $x = 1, 2$ and 3 successively on both sides of (1), we get $A = 1/2$, $B = -4$ and $C = 9/2$.

$\therefore \qquad I = \displaystyle\int \frac{x^2}{(x - 1)(x - 2)(x - 3)} dx$

$\qquad = \displaystyle\int \left\{ \frac{1}{2} \cdot \frac{1}{(x - 1)} - \frac{4}{(x - 2)} + \frac{9}{2} \cdot \frac{1}{(x - 3)} \right\} dx$

$\qquad = \dfrac{1}{2} \displaystyle\int \frac{1}{(x - 1)} dx - 4 \int \frac{dx}{(x - 2)} + \frac{9}{2} \int \frac{dx}{(x - 3)}$

$\qquad = \dfrac{1}{2} \ln |x - 1| - 4 \ln |x - 2| + \dfrac{9}{2} \ln |x - 3| + c$

Two Important deductions of case I

Deduction I : If everywhere same quantity in the given fraction then put same quantity equal to another quantity for the sake of partial fractions that at last substitute the value of another quantity.

Examples : Evaluate $\int \dfrac{(x^2-4)}{(x^2+1)(x^2+2)(x^2+3)}dx$.

Solution. Let $I = \int \dfrac{(x^2-4)}{(x^2+1)(x^2+2)(x^2+3)}dx$

The given fraction has everywhere x^2. Put $x^2 = t$ for the sake of partial fractions

$\therefore \quad \dfrac{(x^2-4)}{(x^2+1)(x^2+2)(x^2+3)} = \dfrac{(t-4)}{(t+1)(t+2)(t+3)}$

$\qquad \equiv \dfrac{A}{t+1} + \dfrac{B}{t+2} + \dfrac{C}{t+3}$ (say) $\qquad\qquad$ ...(1)

$\therefore \quad A = \lim_{t \to -1}(t+1)\left\{\dfrac{(t-4)}{(t+1)(t+2)(t+3)}\right\}$

$\qquad = \lim_{t \to -1}\left\{\dfrac{(t-4)}{(t+2)(t+3)}\right\}$

$\qquad = \dfrac{-1-4}{(-1+2)(-1+3)} = -\dfrac{5}{2}$

$\qquad B = \lim_{t \to -2}(t+2)\left\{\dfrac{(t-4)}{(t+1)(t+2)(t+3)}\right\}$

$\qquad = \lim_{t \to -2}\left\{\dfrac{(t-4)}{(t+1)(t+3)}\right\} = \dfrac{(-2-4)}{(-2+1)(-2+3)} = 6$

and $\quad C = \lim_{t \to -3}(t+3)\left\{\dfrac{(t-3)}{(t+1)(t+2)(t+3)}\right\}$

$\qquad = \lim_{t \to -3}\left\{\dfrac{(t-4)}{(t+1)(t+2)}\right\} = \dfrac{(-3-4)}{(-3+1)(-3+2)} = -\dfrac{7}{2}$

Substituting the values of A, B and C in (1), we have

$\dfrac{(t-4)}{(t+1)(t+2)(t+3)} \equiv -\dfrac{5}{2}\cdot\dfrac{1}{(t+1)} + \dfrac{6}{(t+2)} - \dfrac{7}{2}\cdot\dfrac{1}{(t+3)}$

or $\quad \dfrac{(x^2-4)}{(x^2+1)(x^2+2)(x^2+3)} \equiv -\dfrac{5}{2(x^2+1)} + \dfrac{6}{(x^2+2)} - \dfrac{7}{2(x^2+3)}$

$\therefore \quad I = \int \dfrac{(x^2-4)}{(x^2+1)(x^2+2)(x^2+3)}dx$

$\qquad = -\dfrac{5}{2}\int \dfrac{dx}{(x^2+1)} + 6\int \dfrac{dx}{(x^2+2)} - \dfrac{7}{2}\int \dfrac{dx}{(x^2+3)}$

$\qquad = -\dfrac{5}{2}\tan^{-1}x + \dfrac{6}{\sqrt{2}}\tan^{-1}\left(\dfrac{x}{\sqrt{2}}\right) - \dfrac{7}{2\sqrt{3}}\tan^{-1}\left(\dfrac{x}{\sqrt{3}}\right) + c$

Deduction II : If $f(x)/g(x)$ is an improper fraction, then after division let remainder $\phi(x)$ has lower degree of $g(x)$.

Let $g(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$

then by actual division $\dfrac{f(x)}{g(x)} = h(x) + \dfrac{\phi(x)}{g(x)}$

$$\equiv h(x) + \frac{A_1}{(x - a_1)} + \frac{A_2}{(x - a_2)} + \ldots + \frac{A_n}{(x - a_n)}$$

Examples : Evaluate : $\displaystyle\int \frac{(x - a)(x - b)(x - c)}{(x - \alpha)(x - \beta)(x - \gamma)}\,dx$.

Solution. Let $I = \displaystyle\int \frac{(x - a)(x - b)(x - c)}{(x - \alpha)(x - \beta)(x - \gamma)}\,dx$

The given fraction is an improper fraction then by actual division, since same degree in above and below.

i.e., $\quad \dfrac{x.x.x}{x.x.x} = 1$

then $\quad \dfrac{(x - a)(x - b)(x - c)}{(x - \alpha)(x - \beta)(x - \gamma)} = 1 + \dfrac{f(x)}{(x - \alpha)(x - \beta)(x - \gamma)}$

where $f(x)$ is a polynomial of second degree

$\therefore \quad \dfrac{(x - a)(x - b)(x - c)}{(x - \alpha)(x - \beta)(x - \gamma)} \equiv 1 + \dfrac{A}{(x - \alpha)} + \dfrac{B}{(x - \beta)} + \dfrac{C}{(x - \gamma)}$ (say) $\quad \ldots(1)$

$\therefore \quad A = \displaystyle\lim_{x \to \alpha}(x - \alpha)\left\{\frac{(x - a)(x - b)(x - c)}{(x - \alpha)(x - \beta)(x - \gamma)}\right\}$

$= \displaystyle\lim_{x \to \alpha}\left\{\frac{(x - a)(x - b)(x - c)}{(x - \beta)(x - \gamma)}\right\} = \frac{(\alpha - a)(\alpha - b)(\alpha - c)}{(\alpha - \beta)(\alpha - \gamma)}$

$B = \displaystyle\lim_{x \to \beta}(x - \beta)\left\{\frac{(x - a)(x - b)(x - c)}{(x - \alpha)(x - \beta)(x - \gamma)}\right\} = \displaystyle\lim_{x \to \beta}\left\{\frac{(x - a)(x - b)(x - c)}{(x - \alpha)(x - \gamma)}\right\} = \frac{(\beta - a)(\beta - b)(\beta - c)}{(\beta - \alpha)(\beta - \gamma)}$

and $\quad C = \displaystyle\lim_{x \to \gamma}(x - \gamma)\left\{\frac{(x - a)(x - b)(x - c)}{(x - \alpha)(x - \beta)(x - \gamma)}\right\}$

$= \displaystyle\lim_{x \to \gamma}\left\{\frac{(x - a)(x - b)(x - c)}{(x - \alpha)(x - \beta)}\right\} = \frac{(\gamma - a)(\gamma - b)(\gamma - c)}{(\gamma - \alpha)(\gamma - \beta)}$

Substituting the values of A, B and C in (1), we have

$$\frac{(x - a)(x - b)(x - c)}{(x - \alpha)(x - \beta)(x - \gamma)} \equiv 1 + \frac{(\alpha - a)(\alpha - b)(\alpha - b)}{(\alpha - \beta)(\alpha - \gamma)} \cdot \frac{1}{(x - \alpha)}$$

$$+ \frac{(\beta - a)(\beta - b)(\beta - c)}{(\beta - \alpha)(\beta - \gamma)} \cdot \frac{1}{(x - \beta)} + \frac{(\gamma - a)(\gamma - b)(\gamma - c)}{(\gamma - \alpha)(\gamma - \beta)} \cdot \frac{1}{(x - \gamma)}$$

$$\therefore \quad I = \int \frac{(x-\alpha)(x-b)(x-c)}{(x-\alpha)(x-\beta)(x-\gamma)} dx$$

$$= \int \left\{ 1 + \frac{(\alpha-a)(\alpha-b)(\alpha-c)}{(\alpha-\beta)(\alpha-\gamma)} \cdot \frac{1}{(x-\alpha)} - \frac{(\beta-\alpha)(\beta-b)(\beta-c)}{(\beta-\alpha)(\beta-\gamma)} \cdot \frac{1}{(x-\beta)} \right\} dx$$

$$= x + \frac{(\alpha-a)(\alpha-b)(\alpha-c)}{(\alpha-\beta)(\alpha-\gamma)} \ln|x-\alpha| + \frac{(\beta-\alpha)(\beta-b)(\beta-c)}{(\beta-\alpha)(\beta-\gamma)} \ln|x-\beta|$$

$$+ \frac{(\gamma-a)(\gamma-b)(\gamma-c)}{(\gamma-\alpha)(\gamma-\beta)} \ln|x-\gamma| + c$$

Case II : Integration of fractions with repeated linear factors in the denominator

A repeated linear factor $(x-a)^n$ of denominator then corresponds partial fractions of the form

$$\frac{f(x)}{(x-a)^n} \equiv \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \dots + \frac{A_n}{(x-a)^n}$$

where $A_1, A_2, A_3, \dots, A_n$ are constants, can be determined equating the numerator of L.H.S. to the numerator of R.H.S. (after L.C.M.) and substituting $x = a$, we get A_1. e.g., $\frac{(2x+7)}{(x-1)^2(x+5)}$ will be partial

fractions of the form $\frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+5)}$.

Examples : Evaluate $\int \frac{(2x-5)}{(x+3)(x+1)^2} dx$

Solution. Let $I = \int \frac{(2x-5)}{(x+3)(x+1)^2} dx$

Here $\frac{(2x-5)}{(x+3)(x+1)^2} \equiv \frac{A}{(x+3)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$ (say) ...(1)

$$\Rightarrow \quad (2x-5) \equiv A(x+1)^2 + B(x+1)(x+3) + C(x+3) \qquad ...(2)$$

Putting $x = -1$ and $x = -3$ in (2), we get

$$A = -\frac{11}{4}, \quad C = -\frac{7}{2}.$$

Equating the coefficients of x^2 on the sides of (2), we get $0 = A + B$

$$\therefore \quad B = \frac{11}{4}$$

Substituting the values of A, B and C in (1), we have

$$\frac{(2x-5)}{(x+3)(x+1)^2} \equiv -\frac{11}{4(x+3)} + \frac{11}{4(x+1)} - \frac{7}{2(x+1)^2}$$

$$\therefore \quad I = \int \frac{(2x-5)}{(x+3)(x+1)^2} dx$$

$$= -\frac{11}{4} \int \frac{dx}{(x+3)} + \frac{11}{4} \int \frac{dx}{(x+1)} - \frac{7}{2} \int \frac{dx}{(x+1)^2}$$

$$= -\frac{11}{4} \ln|x+3| + \frac{11}{4} \ln|x+1| + \frac{7}{2(x+1)} + c$$

Case III: Integration of fractions with non repeated quadratic factors in the denominator

To every quadratic factor (which can not be factorized into linear factors) of the form $ax^2 + bx + c$ in the denominator, there will be partial fraction of the form $\dfrac{Ax + B}{ax^2 + bx + c}$ where A and B are constants to be determined.

e.g., $\dfrac{(5x^2 + 3x - 8)}{(x^2 + 5x + 7)(x - 2)(x + 4)} \equiv \dfrac{Ax + B}{(x^2 + 5x + 7)} + \dfrac{C}{(x - 2)} + \dfrac{D}{(x + 4)}$

Examples : Evaluate : $\displaystyle\int \dfrac{(2x^2 - 11x + 5)}{(x - 3)(x^2 + 2x + 5)}\,dx$.

Solution. Let $I = \displaystyle\int \dfrac{(2x^2 - 11x + 5)}{(x - 3)(x^2 + 2x + 5)}\,dx$

Here $\dfrac{(2x^2 - 11x + 5)}{(x - 3)(x^2 + 2x + 5)} \equiv \dfrac{A}{(x - 3)} + \dfrac{Bx + C}{(x^2 + 2x + 5)}$ (say) ...(1)

$\Rightarrow \quad (2x^2 - 11x + 5) = A(x^2 + 2x + 5) + (Bx + C)(x - 3)$ ...(2)

Putting $x - 3 = 0$ or $x = 3$ in (2), we get

$$A = -\dfrac{1}{2}$$

Equating the coefficients of x^2 and x in (2), we have

$$2 = A + B, \ -11 = 2A - 3B + C$$

then $B = \dfrac{5}{2}$ and $C = -\dfrac{5}{2}$.

Substituting the values of A, B and C in (1), we have

$$\dfrac{2x^2 - 11x + 5}{(x - 3)(x^2 + 2x + 5)} \equiv -\dfrac{1}{2(x - 3)} + \dfrac{(5x - 5)}{2(x^2 + 2x + 5)}$$

Examples : Evaluate : $\displaystyle\int \dfrac{dx}{(3x^2 + 2x + 1)}$.

Solution. Let $I = \displaystyle\int \dfrac{dx}{(3x^2 + 2x + 1)}$

$= \dfrac{1}{3}\displaystyle\int \dfrac{dx}{\left(x^2 + \dfrac{2}{3}x + \dfrac{1}{3}\right)} = \dfrac{1}{3}\displaystyle\int \dfrac{dx}{\left(x + \dfrac{1}{3}\right)^2 - \dfrac{1}{9} + \dfrac{1}{3}}$

$= \dfrac{1}{3}\displaystyle\int \dfrac{dx}{\left\{\left(x + \dfrac{1}{3}\right)^2 + \left(\dfrac{\sqrt{2}}{3}\right)^2\right\}}$

$= \dfrac{1}{3} \cdot \dfrac{1}{\left(\dfrac{\sqrt{2}}{3}\right)} \tan^{-1}\left(\dfrac{3x + 1}{\sqrt{2}}\right) + c$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + c$$

(ii) $\int \dfrac{(px+q)}{(ax^2+bx+c)} dx.$

Rule : Here we express the numerator as follows :

Numerator = l (Differential coefficient of Denominator) + m where l and m are constants.

i.e., $px + q = l\left\{\dfrac{d}{dx}(ax^2+bx+c)\right\} + m$

or $px + q = l(2ax+b) + m$

Comparing the coefficient of x and constant terms on both sides, we get

$p = 2al$ and $q = bl + m$

∴ $l = \dfrac{p}{2a}$ and $m = \left(q - \dfrac{pb}{2a}\right)$ ∴ $px + q = \dfrac{p}{2a}(2ax+b) + \left(q - \dfrac{pb}{2a}\right)$

Thus, we get $\int \dfrac{px+q}{(ax^2+bx+c)} dx = \int \dfrac{\dfrac{p}{2a}(2ax+b) + \left(q - \dfrac{pb}{2a}\right)}{(ax^2+bx+c)} dx$

$$= \frac{p}{2a} \int \frac{(2ax+b)}{(ax^2+bx+c)} dx + \left(q - \frac{pb}{2a}\right)\int \frac{dx}{(ax^2+bx+c)}$$

$$= \frac{p}{2a} \ln |ax^2 + bx + c| + \left(q - \frac{pb}{2a}\right)\int \frac{dx}{(ax^2+bx+c)},$$

the integral on R.H.S. can be evaluated easily as in (i).

Examples : Evaluate : $\int \dfrac{(5x-2)}{(3x^2+2x+1)} dx.$

Solution. Let $I = \int \dfrac{(5x-2)}{(3x^2+2x+1)} dx$ ...(1)

∵ $\dfrac{d}{dx}(3x^2+2x+1) = 6x+2$

so we write $5x - 2 = l(6x+2) + m$

Comparing the coefficient of x and constant terms, we get

$5 = 5l$ and $-2 = 2l + m$

∴ $l = \dfrac{5}{6}$ and $m = -\dfrac{11}{3}$

∴ $I = \int \dfrac{5x-2}{(3x^2+2x+1)} dx = \int \dfrac{\dfrac{5}{6}(6x+2) - \dfrac{11}{3}}{(3x^2+2x+1)} dx$

$$= \frac{5}{6} \int \frac{(6x+2)dx}{(3x^2+2x+1)} - \frac{11}{3} \int \frac{dx}{(3x^2+2x+1)}$$

$$= \frac{5}{6} \ln |3x^2 + 2x + 1| - \frac{11}{9} \int \frac{dx}{\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2}$$

$$= \frac{5}{6} \ln |3x^2 + 2x + 1| - \frac{11}{3\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + c$$

(iii) $\int \frac{(px^2+qx+r)}{(ax^2+bx+c)} dx.$

Rule : Here express the numerator as follows :

Numerator = l (Denominator) + m (Differential coefficient of Denominator) + n

where l, m, n are constants.

i.e., $px^2 + qx + r = l (ax^2 + bx + c) + m\left\{\frac{d}{dx}(ax^2+bx+c)\right\} + n$

or $px^2 + qx + r = l (ax^2 + bx + c) + m(2ax + b) + n$

Comparing the coefficients of x^2, x and constant terms on both sides, we have

$p = la$, $q = bl + 2am$ and $r = lc + mb + n$

then we get

and $n = \frac{2a^2r - 2apc - abq + b^2}{2a^2}$ ...(1)

Then, we get $\int \frac{(px^2+qx+r)}{(ax^2+bx+c)}dx = \int \frac{l(ax^2+bx+c)+m(2ax+b)}{(ax^2+bx+c)}dx$

$$= l \int dx + m \int \frac{(2ax+b)}{(ax^2+bx+c)}dx + n \int \frac{dx}{(ax^2+bx+c)}$$

$$= ln + m \ln | ax^2 + bx + c | + n \int \frac{dx}{ax^2+bx+c}$$

the integral on R.H.S. can be evaluated easily and in last substitute the values of l, m & n from (1).

Example : Evaluate : $\int \frac{(2x^2+3x+4)}{(x^2+6x+10)} dx.$

Solution. Let I = $\int \frac{(2x^2+3x+4)}{(x^2+6x+10)} dx$

We write $2x^2 + 3x + 4 = l(x^2 + 6x + 10) + m\frac{d}{dx}(x^6 + 6x + 10) + n$

$= l(x^2 + 6x + 10) + m(2x + 6) + n$

Comparing the coefficients of x^2, x and constant terms on both sides then we get

$2 = l$, $3 = 6l + 2m$ and $4 = 10l + 6m + n$

$\therefore$ $l = 2$, $m = -\frac{9}{2}$ and $n = 11.$

$\therefore \qquad I = \int \dfrac{(2x^2 + 3x + 4)}{(x^2 + 6x + 10)} dx$

$\qquad = \int \dfrac{2(x^2 + 6x + 10) - \dfrac{9}{2}(2x + 6) + 11}{(x^2 + 6x + 10)} dx$

$\qquad = \int 2\,dx - \dfrac{9}{2}\int \dfrac{(2x + 6)}{(x^2 + 6x + 10)} dx + 11\int \dfrac{dx}{(x^2 + 6x + 10)}$

$\qquad = 2\int dx - \dfrac{9}{2}\int \dfrac{(2x + 6)}{(x^2 + 6x + 10)} dx + 11\int \dfrac{dx}{(x + 3)^2 + 1}$

$\qquad = 2x - \dfrac{9}{2} \ln |\, x^2 + 6x + 10 \,| + 11\, \tan^{-1}(x + 3) + c$

(iv) $\int \dfrac{(p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n)}{(ax^2 + bx + c)} dx \; (n > 2).$

Rule : Here divide the numerator by denominator and express it as

$\qquad$ Quotient $+ \dfrac{\text{remainder}}{ax^2 + bx + c}$

or $\qquad \phi(x) + \dfrac{(px + q)}{(ax^2 + bx + c)}$

where $\phi(x)$ will consist of certain terms which we shall integrate by power formula and

$\dfrac{px + q}{(ax^2 + bx + c)}$ will be integrated as discussed in (ii).

Examples : Evaluate $\int \dfrac{x^4 + 2x^3 + 3x + 1}{x^2 + x + 1} dx.$

Solution. Let $I = \int \dfrac{(x^4 + 2x^3 + 3x + 1)}{(x^2 + x + 1)} dx$

Now,

```
x² + x + 1 | x⁴ + 2x³ + 3x + 1 | x² + x − 2
             x⁴ +  x³ +  x²
             −    −    −
           ─────────────────
             x³ − x² + 3x + 1
             x³ + x² + x
             −    −    −
           ─────────────────
               − 2x² + 2x + 1
               − 2x² − 2x − 2
               +    +    +
             ───────────────
                     4x + 3
             ───────────────
```

$\therefore \qquad \dfrac{x^4 + 2x^3 + 3x + 1}{x^2 + x + 1} = (x^2 + x - 2) + \dfrac{4x + 3}{(x^2 + x + 1)}$

$\therefore \qquad I = \int \dfrac{(x^4 + 2x^3 + 3x + 1)}{(x^2 + x + 1)} dx$

$\qquad = \int (x^2 + x - 2) dx + \int \dfrac{(4x + 3)}{(x^2 + x + 1)} dx$

$\qquad = \int (x^2 + x - 2) dx + \int \dfrac{2(2x + 1) + 1}{(x^2 + x + 1)} dx$

$\qquad = \int (x^2 + x - 2) dx + 2\int \dfrac{(2x + 1)}{(x^2 + x + 1)} dx + \int \dfrac{dx}{\left(x + \dfrac{1}{2}\right)^2 + \left(\dfrac{\sqrt{3}}{2}\right)^2}$

$\qquad = \dfrac{x^3}{3} + \dfrac{x^2}{2} - 2x + 2 \, \ln |\, x^2 + x + 1 \,| + \dfrac{2}{\sqrt{3}} \tan^{-1}\left(\dfrac{2x + 1}{\sqrt{3}}\right) + c$

3. Integrals of the form :

(i) $\quad \displaystyle\int \dfrac{(x^2 + a^2) dx}{(x^4 + kx^2 + a^4)}$

(ii) $\quad \displaystyle\int \dfrac{(x^2 - a^2) dx}{(x^4 + kx^2 + a^4)}$

(iii) $\quad \displaystyle\int \dfrac{1}{(x^4 + kx^2 + a^4)} dx$

(iv) $\quad \displaystyle\int \dfrac{x^2 dx}{(x^4 + kx^2 + a^4)}$ where k is any constant

(i) $\displaystyle\int \dfrac{(x^2 + a^2) dx}{(x^4 + kx^2 + a^4)} dx$.

Rule : Divide above and below by x^2, then

$$\int \dfrac{(x^2 + a^2)}{(x^4 + kx^2 + a^4)} dx = \int \dfrac{\left(1 + \dfrac{a^2}{x^2}\right) dx}{\left(x^2 + k + \dfrac{a^4}{x^2}\right)}$$

Here integral of the numerator $= x - \dfrac{a^2}{x}$

$$= \int \dfrac{\left(1 + \dfrac{a^2}{x^2}\right) dx}{\left(x - \dfrac{a^2}{x}\right)^2 + k + 2a^2}$$

put $\quad x - \dfrac{a^2}{x} = t \qquad\qquad \therefore \quad \left(1 + \dfrac{a^2}{x^2}\right) dx = dt$

then $\quad \displaystyle\int \dfrac{(x^2 + a^2)}{(x^4 + kx^2 + a^4)} dx = \int \dfrac{dt}{t^2 + \sqrt{(k + 2a^2)^2}}$

which is of the form $\int \dfrac{dx}{x^2-a^2}$ or $\int \dfrac{dx}{x^2+a^2}$ and can be integrated.

Examples : Evaluate : $\int \left(\dfrac{x^2+1}{x^4+1}\right)dx$.

Solution. Let $I = \int \dfrac{x^2+1}{x^4+1}dx$

Dividing the numerator and denominator by x^2, we get

$$I = \int \dfrac{1+\dfrac{1}{x^2}}{x^2+\dfrac{1}{x^2}}dx = \int \dfrac{1+\dfrac{1}{x^2}}{\left(x-\dfrac{1}{x}\right)^2+2}dx$$

Put $\quad x-\dfrac{1}{x} = t$, so that $\left(1+\dfrac{1}{x^2}\right)dx = dt$.

$$\therefore \quad I = \int \dfrac{dt}{t^2+(\sqrt{2})^2} = \dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{t}{\sqrt{2}}\right)+C$$

$$= \dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{x-1/x}{\sqrt{2}}\right)+C = \dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{x^2-1}{\sqrt{2}x}\right)+C.$$

Examples : Evaluate : $\int \dfrac{dx}{x(x^5+1)}$

Solution. Let $I = \int \dfrac{dx}{x(x^5+1)}$

$$= \int \dfrac{dx}{x^6\left(1+\dfrac{1}{x^5}\right)} = \int \dfrac{dx}{x^6(1+x^{-5})}$$

Put $\quad 1+x^{-5} = t$

$$\therefore \quad -5x^{-6}\,dx = dt \qquad \text{or} \qquad \dfrac{dx}{x^6} = -\dfrac{dt}{5}$$

then $\quad I = -\dfrac{1}{5}\int \dfrac{dt}{t} = -\dfrac{1}{5}\ln|t|+c$

$$= -\dfrac{1}{5}\ln|1+x^{-5}|+c$$

(ii) $\int \dfrac{dx}{x^2(x^n+1)^{(n-1)/n}}$.

Rule : Taking x^n common and put $1 + x^{-n} = t$

$\therefore \quad \displaystyle\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}} = \int \frac{dx}{x^2 \cdot x^{n-1}\left(1+\dfrac{1}{x^n}\right)^{(n-1)/n}}$

$= \displaystyle\int \frac{dx}{x^{n+1}(1+x^{-n})^{(n-1)/n}}$

Put $\quad 1 + x^{-n} = t$

$\therefore \quad -nx^{-n-1}\,dx = dt \text{ or } \dfrac{dx}{x^{n+1}} = -\dfrac{dt}{n}$

then $\quad \displaystyle\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}} = -\frac{1}{n}\int \frac{dt}{t^{(n-1)/n}}$

$= -\dfrac{1}{n}\displaystyle\int t^{1/n-1}\,dt$

$= -\dfrac{1}{n}\cdot\dfrac{t^{1/n-1+1}}{1/n-1+1} + c$

$= -t^{1/n} + c$

$= -(1 + x^{-n})^{1/n} + c$

In 1st integral put $x^2 + 2x + 3 = t$ and in second integral put $x + 1 = u$

$\therefore \quad (2x + 2)dx = dt \text{ and } dx = du$

then $\quad I = \displaystyle\int \frac{dt}{t^2} + \int \frac{du}{(u^2+2)^2}$

$= -\dfrac{1}{t} + \left\{\dfrac{u}{4(u^2+2)} + \dfrac{1}{4}\displaystyle\int \dfrac{du}{u^2+2}\right\}$

$= -\dfrac{1}{t} + \dfrac{u}{4(u^2+2)} + \dfrac{1}{4\sqrt{2}}\tan^{-1}\left(\dfrac{u}{\sqrt{2}}\right) + c$

Hence $I = -\dfrac{1}{(x^2+2x+3)} + \dfrac{(x+1)}{4(x^2+2x+3)} + \dfrac{1}{4\sqrt{2}}\tan^{-1}\left(\dfrac{x+1}{\sqrt{2}}\right) + c$

IV. Integration of Irrational Functions

1. Integration by Rationalisation :

Rule : Multiplying the numerator and the denominator by the same quantity then the numerator or denominator transforms the irrational function into any of the standard results.

Examples : Evaluate : $\displaystyle\int \sqrt{\left(\dfrac{1-x}{1+x}\right)}\,dx.$

Solution. Let $I = \displaystyle\int \sqrt{\left(\dfrac{1-x}{1+x}\right)}\,dx$

$= \displaystyle\int \frac{\sqrt{1-x}}{\sqrt{1+x}}\,dx$

Multiplying above and below by $\sqrt{(1-x)}$, we get

$$I = \int \frac{\sqrt{(1-x)} \cdot \sqrt{(1-x)}}{\sqrt{(1+x)} \cdot \sqrt{(1-x)}} dx$$

$$= \int \frac{(1-x)}{\sqrt{(1-x^2)}} dx$$

$$= \int \frac{dx}{\sqrt{(1-x^2)}} + \int \frac{-x\,dx}{\sqrt{(1-x^2)}}$$

in second integral on R.H.S. put $1 - x^2 = t^2$

$\therefore \quad - x\,dx = t\,dt$

then $\quad I = \int \frac{dx}{\sqrt{(1-x^2)}} + \int \frac{t\,dt}{t}$

$$= \int \frac{dx}{\sqrt{(1-x^2)}} + \int dt$$

$$= \sin^{-1} x + t + c$$

$$= \sin^{-1} x + \sqrt{(1-x^2)} + c$$

2. Integrals of the form

$$\int f\left\{ x, \left(\frac{ax+b}{cx+d}\right)^{m/n}, ...,\left(\frac{ax+b}{cx+d}\right)^{r/s} \right\} dx$$

where f is the rational function of its arguments.

Rule : In this form substitute $\left(\dfrac{ax+b}{cx+d}\right) = t^{\alpha}$

where α is L.C.M. of the denominator of the fractions $\dfrac{m}{n}, ..., \dfrac{r}{s}$ i.e., L.C.M. of n, ..., s.

Examples : Evaluate : $\int \dfrac{dx}{(1+x)^{1/2} - (1+x)^{1/3}}$.

Solution. Let $I = \int \dfrac{dx}{(1+x)^{1/2} - (1+x)^{1/3}}$

Here L.C.M. of 2 and 3 is 6

$\therefore \quad$ put $1 + x = t^6 \quad \therefore \quad dx = 6t^5\,dt$

then $\quad I = \int \dfrac{6t^5\,dt}{(t^3 - t^2)} = 6\int \dfrac{t^3}{(t-1)}\,dt = 6\int \left(t^2 + t + 1 + \dfrac{1}{t-1}\right)dt$

$$= 6\left\{ \frac{t^3}{3} + \frac{t^2}{2} + t + \ell n\,|t-1| \right\} + c$$

$$= 2(1+x)^{1/2} + 3(1+x)^{1/3} + 6(1+x)^{1/6} + 6\,\ell n\,|\,(1+x)^{1/6} - 1\,| + c$$

$$(\because \quad t = (1+x)^{1/6})$$

3. Integrals of the form :

(i) $\quad \int \dfrac{dx}{\sqrt{(ax^2 + bx + c)}}$

(ii) $\quad \sqrt{ax^2 + bx + c} \ dx$

(iii) $\quad \int \dfrac{(px + q)}{\sqrt{(ax^2 + bx + c)}} dx$

(iv) $\quad \int (px + q)\sqrt{(ax^2 + bx + c)} \ dx$

(v) $\quad \int \dfrac{(px^2 + qx + r)}{\sqrt{(ax^2 + bx + c)}} dx$

(vi) $\quad \int (px^2 + qx + r)\sqrt{(ax^2 + bx + c)} \ dx$

(I) $\int \dfrac{dx}{\sqrt{(ax^2 + bx + c)}}$

Rule : We have

$$ax^2 + bx + c = a\left\{ x^2 + \frac{b}{a}x + \frac{c}{a}\right\} = a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right\}$$

$\therefore \quad I = \int \dfrac{dx}{\sqrt{(ax^2 + bx + c)}} = \dfrac{1}{\sqrt{a}} \int \dfrac{dx}{\sqrt{\left\{\left(x + \dfrac{b}{2a}\right)^2 + \dfrac{c}{a} - \dfrac{b^2}{4a^2}\right\}}}$

Case I : If $b^2 - 4ac > 0$ and $a > 0$

then $\quad I = \dfrac{1}{\sqrt{a}} \int \dfrac{dx}{\sqrt{\left(x + \dfrac{b}{2a}\right)^2 - \left(\dfrac{\sqrt{(b^2 - 4ac)}}{2a}\right)^2}}$

which is of the form $\int \dfrac{dx}{\sqrt{(x^2 - a^2)}}$ and can be integrated.

Case II : If a is negative. Let $a = -\lambda$.

$\therefore \quad I = \int \dfrac{dx}{\sqrt{(ax^2 + bx + c)}} = \int \dfrac{dx}{\sqrt{(c + bx - \lambda x^2)}}$

$= \dfrac{1}{\sqrt{\lambda}} \int \dfrac{dx}{\sqrt{\left(\dfrac{c}{\lambda} + \dfrac{b}{\lambda}x - x^2\right)}}$

$= \dfrac{1}{\sqrt{\lambda}} \int \dfrac{dx}{\sqrt{\left\{\left(\dfrac{b^2 + 4c\lambda^2}{4\lambda^2}\right) - \left(x - \dfrac{b}{2\lambda}\right)^2\right\}}}$

$= \dfrac{1}{\sqrt{(-a)}} \int \dfrac{dx}{\sqrt{\left\{\left(\dfrac{b^2 + 4ac}{4a^2}\right) - \left(x - \dfrac{b}{2a}\right)^2\right\}}}$ $\qquad (\sqrt{-a} \text{ is real, } \because a < 0)$

which is of the form $\int \dfrac{dx}{\sqrt{a^2 - x^2}}$ and can be integrated.

Examples : Evaluate : $\int \dfrac{dx}{\sqrt{(x^2 + x - 2)}}$

Solution. Let $I = \int \dfrac{dx}{\sqrt{(x^2 + x - 2)}}$

$$= \int \dfrac{dx}{\sqrt{\left(x + \dfrac{1}{2}\right)^2 - \left(\dfrac{3}{2}\right)^2}}$$

We know $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right|$

$$\int \dfrac{dx}{\sqrt{(x^2 + x - 2)}} = \ln\left|\left(x + \dfrac{1}{2}\right) + \sqrt{\left(x + \dfrac{1}{2}\right)^2 - \dfrac{9}{4}}\right|$$

(v) $\int \dfrac{(px^2 + qx + r)}{\sqrt{(ax^2 + bx + c)}}dx.$

Rule : We can express the numerator as follows :

$$px^2 + qx + r = l(ax^2 + bx + c) + m\dfrac{d}{dx}(ax^2 + bx + c) + n$$

or $\quad px^2 + qx + r = l(ax^2 + bx + c) + m(2ax + b) + n$

where l, m and n are constants

Comparing the coefficients of x^2, x and constant terms on both sides, we get

$\quad p = al$, $q = bl + 2am$ and $r = lc + bm + n$

$\therefore \quad l = \dfrac{p}{a}$, $m = \dfrac{1}{2a}\left(q - \dfrac{bp}{a}\right)$ and $n = r - \dfrac{pc}{a} + \dfrac{b}{2a}\left(q - \dfrac{bp}{a}\right)$ ...(1)

$\therefore \quad I = \int \dfrac{(px^2 + qx + r)}{\sqrt{(ax^2 + bx + c)}}dx$

$$= \int \dfrac{l(ax^2 + bx + c) + m(2ax + b) + n}{\sqrt{(ax^2 + bx + c)}}dx$$

$$= l\int \sqrt{(ax^2 + bx + c)}\,dx + m\int \dfrac{(2ax + b)}{\sqrt{ax^2 + bx + c}}dx + n\int \dfrac{dx}{\sqrt{(ax^2 + bx + c)}}$$

$$= l\int \sqrt{(ax^2 + bx + c)}\,dx + 2m\sqrt{(ax^2 + bx + c)} + \int \dfrac{dx}{\sqrt{(ax^2 + bx + c)}} \quad\text{...(2)}$$

The integrals on R.H.S. can be evaluated easily as in (i) and (ii) and at last substituting the values of l, m and n from (1) in (2).

Examples : Evaluate : $\int \dfrac{(x^2 - x + 1)}{\sqrt{(2x^2 - x + 2)}} dx.$

Solution. Let $I = \int \dfrac{(x^2 - x + 1)}{\sqrt{(2x^2 - x + 2)}}$

Here $\quad x^2 - x + 1 = l(2x^2 - x + 2) + m\dfrac{d}{dx}(2x^2 - x + 2) + n$

or $\quad x^2 - x + 1 = l(2x^2 - x + 2) + m(4x - 1) + n$

Comparing the coefficients of x^2, x and constant terms on both sides, then

$\quad 1 = 2l, \quad -1 = -l + 4m \quad \text{and} \quad 1 = 2l - m + n$

$\therefore \quad l = \dfrac{1}{2},\ m = -\dfrac{1}{8}\ \text{and}\ n = -\dfrac{1}{8}$

(i) $\int \dfrac{dx}{(px + q)\sqrt{(ax + b)}}$.

Rule : Here we put $ax + b = t^2$

Note :

The above rule will be applicable if the numerator is the function of x and f(x) in places of unity

i.e., $\quad \int \dfrac{f(x)dx}{(px + q)\sqrt{(ax + b)}}$ will be evaluated by the above rule.

Examples : Evaluate : $\int \dfrac{(8x^2 + 26x + 16)}{(4x + 3)\sqrt{(2x + 5)}} dx$.

Solution. Let $I = \int \dfrac{(8x^2 + 26x + 16)}{(4x + 3)\sqrt{(2x + 5)}} dx$

Here $8x^2 + 26x + 16 = (4x + 3)(2x + 5) + 1$

then $\quad I = \int \dfrac{(4x + 3)(2x + 5) + 1}{(4x + 3)\sqrt{(2x + 5)}} dx$

$\qquad = \int \sqrt{(2x + 5)}\, dx + \int \dfrac{dx}{(4x + 3)\sqrt{(2x + 5)}}$

In second integral on R.H.S. put $2x + 5 = t^2$

$\therefore \quad 2dx = 2t\, dt \qquad \text{or} \qquad dx = t\, dt$

$\therefore \quad I = \int (2x + 5)^{1/2} dx + \int \dfrac{t\, dt}{t(2t^2 - 7)}$

$\qquad = \int (2x + 5)^{1/2} dx + \dfrac{1}{2}\int \dfrac{dt}{t^2 - (\sqrt{7}/2)^2}$

$$= \frac{2}{3} \cdot \frac{1}{2}(2x+5)^{3/2} + \frac{1}{2} \cdot \ln\left|t + \sqrt{(t^2 - (\sqrt{7/2})^2)}\right| + c$$

$$= \frac{1}{3}(2x+5)^{3/2} + \frac{1}{2}\ln\left|\sqrt{(2x+5)} + \sqrt{(2x+3/2)}\right| + c$$

(ii) $\int \dfrac{dx}{(px^2 + qx + r)\sqrt{(ax+b)}}$.

Rule : Here we put $ax + b = t^2$

Note : The above rule will be applicable if the numerator is the function of x say f(x) in place of unity

i.e., $\int \dfrac{f(x)}{(px^2 + qx + r)\sqrt{(ax+b)}}\, dx$ will be evaluated by the above rule.

(iii) $\int \dfrac{dx}{(px + q)\sqrt{(ax^2 + bx + c)}}$.

Rule : Here we put $px + q = \dfrac{1}{t}$

Note : The above rule will be applicable if the numerator is the function of x say f(x) in place of unity.

i.e., $\int \dfrac{f(x)dx}{(px + q)\sqrt{(ax^2 + bx + c)}}$ will be evaluated by the above rule.

Examples : Evaluate :

$$\int \frac{dx}{(x-a)^{3/2}(x+a)^{1/2}} = \int \frac{dx}{(x-a)\sqrt{(x^2 - a^2)}}.$$

Solution. Let I = $\int \dfrac{dx}{(x-a)\sqrt{(x^2 - a^2)}}$

Put $x - a = \dfrac{1}{t}$ $\therefore$ $dx = -\dfrac{1}{t^2}dt$

then   I = $\int \dfrac{-\dfrac{1}{t^2}dt}{\dfrac{1}{t}\sqrt{\left\{\left(a+\dfrac{1}{t}\right)^2 - a^2\right\}}}$

$$= -\int \frac{dt}{\sqrt{(at+1)^2 - a^2 t^2}}$$

$$= -\int \frac{dt}{\sqrt{(2at+1)}}$$

$$= -\frac{2}{2a}\sqrt{(2at+1)} + c$$

$$= -\frac{1}{a}\sqrt{\left(\frac{2a}{x-a} + 1\right)} + c \qquad \left(\because t = \frac{1}{x-a}\right)$$

$$= -\frac{1}{a}\sqrt{\left(\frac{x+a}{x-a}\right)} + c$$

(iv) $\int \dfrac{dx}{(px+q)^r\sqrt{(ax^2+bx+c)}}$.

Rule : Here we put $px + q = \dfrac{1}{t}$

Note : The above rule will be applicable if the numerator is the function of x say f(x) in place of unity

i.e., $\int \dfrac{f(x)dx}{(px+q)^r\sqrt{(ax^2+bx+c)}}$ will be evaluated by the above rule.

Examples : Evaluate : $\int \dfrac{dx}{(x-2)^2\sqrt{(x^2-4x+5)}}$.

Solution. Let $I = \int \dfrac{dx}{(x-2)^2\sqrt{(x^2-4x+5)}}$

Put $\quad x - 2 = \dfrac{1}{t}$

$\therefore \qquad dx = -\dfrac{1}{t^2}\,dt$

then $\quad I = \int \dfrac{-\dfrac{1}{t^2}dt}{\dfrac{1}{t^3}\sqrt{\left\{\left(\dfrac{1}{t}\right)^2 + 1\right\}}} = -\int \dfrac{t^2}{\sqrt{(1+t^2)}}\,dt$

$$= \int \dfrac{(1+t^2)-1}{\sqrt{(1+t^2)}}\,dt$$

$$= -\int \sqrt{(1+t^2)}\,dt + \int \dfrac{dt}{\sqrt{(1+t^2)}}$$

$$= -\left\{\dfrac{t}{2}\sqrt{(1+t^2)} + \dfrac{1}{2}ln\left|t+\sqrt{(1+t^2)}\right|\right\} + ln\,|\,t + \sqrt{1+t^2}\,| + c$$

$$= -\dfrac{t}{2}\sqrt{(1+t^2)} + \dfrac{1}{2}ln\,|\,t + \sqrt{(1+t^2)}\,| + c$$

$$= -\frac{1}{2(x-2)}\sqrt{\left\{1+\frac{1}{(x+2)^2}\right\}} + \frac{1}{2}\,ln\left|\frac{1}{(x-2)} + \sqrt{1+\left(\frac{1}{x-2}\right)^2}\right| + c$$

$$= -\frac{\sqrt{(x^2-4x+5)}}{2(x-2)^2} + \frac{1}{2}\,ln\left|\frac{1+\sqrt{(x^2-4x+5)}}{(x-2)}\right| + c$$

(v) $\int \dfrac{dx}{(px^2+q)\sqrt{(ax^2+b)}}$.

Rule : Let $I = \int \dfrac{dx}{(px^2+q)\sqrt{(ax^2+b)}}$

Put $x = \dfrac{1}{t}$ $\qquad \therefore \qquad dx = -\dfrac{1}{t^2}dt$

then $I = -\int \dfrac{t\,dt}{(p+qt^2)\sqrt{(a+bt^2)}}$

Now put $a + bt^2 = z^2$ $\qquad \therefore \qquad t\,dt = \dfrac{z\,dz}{b}$

$\therefore \qquad I = -\dfrac{1}{b}\int \dfrac{z\,dz}{\left\{p+q\left(\dfrac{z^2-a}{b}\right)\right\}z} = -\int \dfrac{dz}{(qz^2+bp-aq)}$

It can be easily evaluated from standard results.

Example : Evaluate : $\int \dfrac{dx}{(1+x^2)\sqrt{(1-x^2)}}$.

Solution. Let $I = \int \dfrac{dx}{(1+x^2)\sqrt{(1-x^2)}}$

Put $x = \dfrac{1}{t}$ $\qquad \therefore \qquad dx = -\dfrac{1}{t^2}dt$

then $I = \int \dfrac{-\dfrac{1}{t^2}dt}{\left(1+\dfrac{1}{t^2}\right)\sqrt{\left(1-\dfrac{1}{t^2}\right)}} = -\int \dfrac{t\,dt}{(1+t^2)\sqrt{(t^2-1)}}$

Again put $t^2 - 1 = z^2$ $\qquad \therefore \qquad t\,dt = z\,dz$

$\therefore \qquad I = -\int \dfrac{z\,dz}{(z^2+2)z} = -\int \dfrac{dz}{z^2+(\sqrt{2})^2}$

$\qquad\qquad = -\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{z}{\sqrt{2}}\right) + c$

$\qquad\qquad = -\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{\left(\dfrac{t^2-1}{2}\right)}\right) + c \qquad\qquad (\because \quad t^2 - 1 = z^2)$

$$= -\frac{1}{\sqrt{2}} \tan^{-1}\left(\sqrt{\left(\frac{1-x^2}{2x^2}\right)}\right) + c \qquad \left(\because \ t = \frac{1}{x}\right)$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x\sqrt{2}}\right) + c$$

(vi) $\displaystyle\int \frac{dx}{(px^2 + qx + r)\sqrt{(ax^2 + bx + c)}}$.

Rule : Here we put $\sqrt{\left(\dfrac{ax^2 + bx + c}{px^2 + qx + r}\right)} = t$

Examples : Evaluate : $\displaystyle\int \frac{dx}{(x^2 - 5x + 6)\sqrt{(x^2 - 6x + 10)}}$.

Solution. Let $I = \displaystyle\int \frac{dx}{(x^2 - 5x + 6)\sqrt{(x^2 - 6x + 10)}}$

$\because \qquad \dfrac{1}{(x^2 - 5x + 6)} = \dfrac{1}{(x-2)(x-3)} \equiv \dfrac{A}{x-2} + \dfrac{B}{x-3}$, we get $A = -1$ and $B = 1$

$\therefore \qquad \dfrac{1}{(x^2 - 5x + 6)} = \dfrac{1}{(x-3)} - \dfrac{1}{(x-2)}$

then $\quad I = \displaystyle\int \left(\frac{1}{x-3} - \frac{1}{x-2}\right) \frac{dx}{\sqrt{(x^2 - 6x + 10)}}$

$$= \int \frac{dx}{(x-3)\sqrt{(x-3)^2 + 1}} - \int \frac{dx}{(x-2)\sqrt{(x-3)^2 + 1}}$$

In 1st integral put $x - 3 = \dfrac{1}{t}$ and in 2nd integral put $x - 2 = \dfrac{1}{u}$ on R.H.S.

Then, we get

$$I = \int \frac{-\dfrac{1}{t^2} dt}{\dfrac{1}{t}\sqrt{\left(\dfrac{1}{t^2} + 1\right)}} + \int \frac{\dfrac{1}{u^2} du}{\dfrac{1}{u}\sqrt{\left(\dfrac{1}{u} - 1\right)^2 + 1}}$$

$$= -\int \frac{dt}{\sqrt{(1 + t^2)}} + \int \frac{du}{\sqrt{(2u^2 - 2u + 1)}}$$

$$= -\int \frac{dt}{\sqrt{(1 + t^2)}} + \frac{1}{\sqrt{2}} \int \frac{du}{\sqrt{\{(u - 1/2)^2 + (1/2)^2\}}}$$

$$= -\ln\left| t + \sqrt{(1+t^2)} \right| + \frac{1}{\sqrt{2}} \ln\left| \left(u - \frac{1}{2}\right) + \sqrt{\left\{\left(u - \frac{1}{2}\right)^2 + \frac{1}{4}\right\}} \right| + c$$

$$= -\ln\left| t + \sqrt{(1+t^2)} \right| + \frac{1}{\sqrt{2}} \ln\left| \frac{(2u-1)}{2} + \sqrt{\left(u^2 - u + \frac{1}{2}\right)} \right| + c$$

$$= \ln\left| \frac{1}{(x-3)} + \frac{\sqrt{(x^2-6x+10)}}{(x-3)} \right| + \frac{1}{\sqrt{2}} \ln\left| \frac{\left(\frac{2}{x-2} - 1\right)}{2} + \sqrt{\left\{\left(\frac{1}{x-2}\right)^2 - \left(\frac{1}{x-2}\right) + \frac{1}{2}\right\}} \right| + c$$

$$\left(\because \quad t = \frac{1}{x-3} \text{ and } u = \frac{1}{x-2} \right)$$

$$= -\ln\left| \frac{1 + \sqrt{(x^2-6x+10)}}{(x-3)} \right| + \frac{1}{\sqrt{2}} \ln\left| \frac{(4-x)}{2(x-2)} + \sqrt{\left\{\frac{x^2-6x+10}{2(x-2)^2}\right\}} \right| + c$$

V. Integration of Trigonometric Functions

1. Integration of the from $\int \sin^m x \cos^n x \, dx$

Case I : If m is odd and n is even positive integer, then put cos x = t

Examples : Evaluate : $\int \sin^3 x \cos^2 x \, dx$

Solution. Let $I = \int \sin^3 x \cos^2 x \, dx$

Put $\quad \cos x = t \qquad\qquad \therefore \quad \sin x \, dx = -dt$

then $\quad I = \int (1 - \cos^2 x) \cos^2 x \sin x \, dx$

$$= \int (1 - t^2) \, t^2 \, (-dt)$$

$$= \int (t^4 - t^2) dt$$

$$= \frac{1}{5} t^5 - \frac{1}{3} t^3 + c$$

$$= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + c \qquad\qquad (\because \quad \cos x = t)$$

Case II : If m is even and n is odd positive integer, then put sin x = t.

Examples : Evaluate : $\int \sin^4 x \cos^5 x \, dx$.

Solution. Let $I = \int \sin^4 x \cos^5 x \, dx$

Put $\quad \sin x = t \qquad\qquad \therefore \quad \cos x \, dx = dt$

then $\quad I = \int t^4 (1 - t^2)^2 \, dt$

$$= \int t^4 \left(1 - 2t^2 + t^4\right) dt$$

$$= \int (t^4 - 2t^6 + t^8) dt$$

$$= \frac{1}{5}t^5 - \frac{2}{7}t^7 + \frac{1}{9}t^9 + c$$

$$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + c$$

Case III : If m and n both are odd positive integers, then

If m > n, put sin x = t

If m < n, put cos x = t

If m = n, put sin x = t or cos x = t

Examples : Evaluate : $\int \sin^5 x \cos^3 x \, dx$.

Solution. Let I = $\int \sin^5 x \cos^3 x \, dx$

Put   sin x = t $\therefore$ cos x dx = dt

then   I = $\int t^5 (1 - t^2) dt$

$$= \int (t^5 - t^7) dt$$

$$= \frac{1}{6}t^6 - \frac{1}{8}t^8 + c$$

$$= \frac{1}{6}\sin^6 x - \frac{1}{8}\sin^8 x + c$$

Case IV : If m + n = –ve and even integer then convert the given integrand in terms of tan x and $\sec^2 x$ then put tan x = t.

Examples : Evaluate : $\int \sin^{(-14/9)}\theta \cos^{(-4/9)}\theta \, d\theta$.

Solution. Let I = $\int \sin^{(-14/9)}\theta \cos^{(-4/9)}\theta \, d\theta$

Here m + n = $-\dfrac{14}{9} - \dfrac{4}{9} = -2$ (–ve)

Since m + n = –ve, convert the given integrand in terms of tan x and $\sec^2 x$

$$I = \int \frac{d\theta}{\sin^{14/9}\theta \cos^{4/9}\theta}$$

then $I = \int \dfrac{d\theta}{\dfrac{\sin^{14/9}\theta}{\cos^{14/9}\theta} \cdot \cos^{4/9}\theta \cos^{14/9}\theta} = \int \dfrac{\sec^2 \theta \, d\theta}{(\tan \theta)^{14/9}}$

Put   tan θ = t $\Rightarrow$ $\sec^2 \theta \, d\theta$ = dt

then $I = \int \dfrac{dt}{t^{14/9}} = \int t^{-14/9} \, dt$

$$= \frac{t^{(-14/9)+1}}{-\dfrac{14}{9}+1} + c = -\frac{9}{5}t^{-5/9} + c$$

$$= -\frac{9}{5}(\tan\theta)^{-5/9} + c$$

Case V : If m and n are small even integers, then convert them in terms of multiple angles by using the formulae

$$\cos^2 x = \frac{1+\cos 2x}{2}; \ \sin^2 x = \frac{1-\cos 2x}{2}; \ \sin x \cos x = \frac{\sin 2x}{2}$$

and $2 \cos x \cos y = \cos(x + y) + \cos (x - y)$

Examples : Evaluate : $\int \sin^4 x \cos^2 x \ dx$

Solution. Let $I = \int \sin^4 x \cos^2 x \ dx$

$$= \int \sin^2 x (\sin^2 x \cos^2 x) dx$$

$$= \int \sin^2 x \left(\frac{\sin^2 2x}{4}\right) dx$$

$$= \frac{1}{4}\int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 4x}{2}\right) dx$$

$$= \frac{1}{16}\int (1-\cos 4x - \cos 2x + \cos 4x \cos 2x) dx$$

$$= \frac{1}{32}\int (2 - 2\cos 4x - 2\cos 2x + 2\cos 4x \cos 2x) \ dx$$

$$= \frac{1}{32}\int (2 - 2\cos 4x - 2\cos 2x + \cos 6x + \cos 2x) \ dx$$

$$= \frac{1}{32}\int (2 - \cos 2x - 2\cos 4x + \cos 6x) \ dx$$

$$= \frac{1}{32}\left\{2x - \frac{\sin 2x}{2} - \frac{\sin 4x}{2} + \frac{\sin 6x}{6}\right\} + c$$

Case VI : If m and n are large even positive integers then change in multiple angles with the help of complex numbers.

If $z = e^{ix} = \cos x + i \sin x \Rightarrow \frac{1}{z} = e^{-ix} = \cos x - i \sin x$

$\therefore$ $2 \cos x = z + \frac{1}{z}$ and $2i \sin x = z - \frac{1}{z}$

In general, If $z^n = \cos nx + i \sin nx$ then $\frac{1}{z^n} = \cos nx - i \sin nx$

Since $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$

Evaluate $I = \int \sin^6 x \cos^4 x \ dx$ ...(1)

From (1), $-2^{10} \sin^6 x \cos^4 x = \left(z - \frac{1}{z}\right)^6 \left(z + \frac{1}{z}\right)^4$

$$= z^{10} - 2z^8 - 3z^6 + 8z^4 + 2z^2 - 12 + \frac{2}{z^2} + \frac{8}{z^4} - \frac{3}{z^6} - \frac{2}{z^8} + \frac{1}{z^{10}}$$

$$= \left(z^{10} + \frac{1}{z^{10}}\right) - 2\left(z^8 + \frac{1}{z^8}\right) - 3\left(z^6 + \frac{1}{z^6}\right) + 8\left(z^4 + \frac{1}{z^4}\right) + 2\left(z^2 + \frac{1}{z^2}\right) - 12$$

$$= 2\cos 10x - 2 \cdot 2\cos 8x - 3 \cdot 2\cos 6x + 8 \cdot 2\cos 4x + 2 \cdot 2\cos 2x - 12$$

$$= 2\{\cos 10x - 2\cos 8x - 3\cos 6x + 8\cos 4x + 2\cos 2x - 6\}$$

or $$\sin^6 x \cos^4 x = -\frac{1}{2^9}\{\cos 10x - 2\cos 8x - 3\cos 6x + 8\cos 4x + 2\cos 2x - 6\}$$

$\therefore$ $$I = \int \sin^6 x \cos^4 x \, dx$$

$$= -\frac{1}{2^9}\left\{\int \cos 10x \, dz - 2\int \cos 8x \, dx - 3\int \cos 6x \, dx + 8\int \cos 4x \, dx + 2\int \cos 2x \, dx - 6\int dx\right\}$$

$$= -\frac{1}{2^9}\left\{\frac{\sin 10x}{10} - \frac{2\sin 8x}{8} - \frac{3\sin 6x}{6} + \frac{8\sin 4x}{4} + \frac{2\sin 2x}{2} - 6x\right\} + c$$

Hence $$I = -\frac{1}{2^9}\left\{\frac{\sin 10x}{10} - \frac{\sin 8x}{4} - \frac{\sin 6x}{2} + 2\sin 4x + \sin 2x - 6x\right\} + c$$

2. Integral of the form :

$$\int \tan^m x \sec^n x \, dx$$

Rule :

(i)   If m is even or odd integer and n is even positive integer than put tan x = t.

(ii) If m is odd positive integer and n $\notin$ even positive integer then put sec x = t.

(iii) If m = 0 and n = 2r + 1 $\forall$ r $\in$ N

then $$\int \sec^{2r+1} x \, dx = \int \sec^{2r-1} x \cdot \sec^2 x \, dx$$

then integrate by parts taking $\sec^2 x$ as second function.

Note : If m $\in$ even positive integer and n $\in$ odd positive integer, then integral is **non-Integrable.**

Example : Evaluate : $\int \tan^2 x \sec^2 x \, dx$.

Solution. By Using Ist rule

Here m = 2 and n = 2

So let tan x = t $\Rightarrow$ $\sec^2 x \, dx = dt$

$\Rightarrow$ $$I = \int t^2 \, dt = \frac{t^3}{3} = \frac{1}{3}\tan^3 x$$

We can also solve it by using integral by parts as taking $\tan^2 x$ is Ist function and $\sec^2 x$ as IInd function.

3. Integral of the form :

$$\int \cot^m x \csc^n x \, dx$$

Rule :

(i)   If m is even or odd integer and n is given positive integer then put cot x = t.

(ii)   If m is odd positive integer and n $\notin$ even positive integer then put cosec x = t

(iii)   If m = 0 and n = 2r + 1 $\forall$ r $\in$ N

then $\int \cos ec^{2r+1} x \, dx = \int \cos ec^{2r-1} x . \cos ec^2 x \, dx$

integrate by parts taking cosec²x as second function.

Note : If m $\in$ even positive integer and n $\in$ odd positive integer then integral is non-integrable.

Examples : Evaluate : $\int \cot^2 x \cos ec^4 x \, dx$.

Solution. Let I = $\int \cot^2 x \cos ec^4 x \, dx$

Here   m = 2 and n = 4

or   I = $\int \cot^2 x . (\cot^2 x + 1) . \cos ec^2 x \, dx$ (since cosec²x = 1 + cot²x)

Put   cot x = t

$\therefore$ – cosec² x dx = dt

$\Rightarrow$ cosec² x dx = – dt

then   I = $\int t^2 (t^2 + 1)(-dt)$

$= \int (-t^2 - t^4) \, dt$

$= -\dfrac{t^3}{3} - \dfrac{t^5}{5} + c$

$= -\dfrac{\cot^3 x}{3} - \dfrac{\cot^5 x}{5} + c$

4. Integrals of the form :

(i) $\int \dfrac{dx}{a + b \sin^2 x}$ (ii) $\int \dfrac{dx}{(a + b \cos^2 x)}$

(iii) $\int \dfrac{dx}{a \cos^2 x + b \sin^2 x}$ (iv) $\int \dfrac{dx}{a + b \sin^2 x + c \cos^2 x}$

(v) $\int \dfrac{dx}{(a \sin x + b \cos x)^2}$ (vi) $\int \dfrac{\phi(\tan x) \, dx}{a \sin^2 x + b \sin x \cos x + c \cos^2 x}$

where a, b, c $\in$ R and not at a time all zero.

Rule : We shall always in such cases divide above and below by cos² x; then put tan x = t i.e.,

sec² x dx = dt then the questions shall reduce to the forms $\int \dfrac{dt}{(at^2 + bt + c)}$ or $\int \dfrac{\phi(t) \, dt}{(at^2 + bt + c)}$

Examples : Evaluate : $\int \dfrac{\cos x}{\cos 3x} \, dx$.

Solution. Let I = $\int \dfrac{\cos x}{\cos 3x} \, dx$

$= \int \dfrac{\cos x}{4 \cos^3 x - 3 \cos x} \, dx$

$$= \int \frac{dx}{(4\cos^2 x - 3)}$$

Divide above and below by $\cos^2 x$, then

$$I = \int \frac{\sec^2 x \, dx}{4 - 3\sec^2 x} = \int \frac{\sec^2 x \, dx}{4 - 3(1 + \tan^2 x)}$$

$$= \int \frac{\sec^2 x \, dx}{1 - 3\tan^2 x}$$

Put $\sqrt{3} \tan x = t$ $\qquad \therefore \qquad \sec^2 x \, dx = \dfrac{dt}{\sqrt{3}}$

$$1 = \int \frac{dt/\sqrt{3}}{1 - t^2} = \frac{1}{\sqrt{3}} \int \frac{dt}{(1+t)(1-t)}$$

$$= \frac{1}{\sqrt{3}} \int \left(\frac{1/2}{(1+t)} + \frac{1/2}{(1-t)} \right) dt$$

$$= \frac{1}{\sqrt{3}} \times \frac{1}{2} \left[\int \frac{dt}{1+t} + \int \frac{dt}{1-t} \right]$$

$$= \frac{1}{2\sqrt{3}} \left[\ln|1+t| - \int -\frac{dt}{1-t} \right]$$

$$= \frac{1}{2\sqrt{3}} \left[\ln|1+t| - \ln|1-t| \right]$$

$$= \frac{1}{2\sqrt{3}} \ln \left| \frac{1+t}{1-t} \right|$$

$$1 = \frac{1}{2\sqrt{3}} \ln \left| \frac{1 + \sqrt{3}\tan x}{1 - \sqrt{3}\tan x} \right|$$

5. Integrals of the form :

(i) $\quad \displaystyle\int \frac{(a\sin x + b)}{(a + b\sin x)^2} \, dx$ $\qquad\qquad$ (ii) $\quad \displaystyle\int \frac{(a\cos x + b)}{(a + b\cos x)^2} \, dx$

(iii) $\quad \displaystyle\int \frac{(a\sin x + b)}{(a + b\sin x)^2} \, dx$

Rule : Divide above and below by $\cos^2 x$, then

Put $a \sec x + b \tan x = t$

Examples : Evaluate : $\displaystyle\int \frac{(2\sin x + 5)}{(2 + 5\sin x)^2} \, dx$

Solution. Let $I = \displaystyle\int \frac{(2\sin x + 5)}{(2 + 5\sin x)^2} \, dx$

Divide above and below by $\cos^2 x$, then

$$I = \int \frac{(2 \sec x \tan x + 5 \sec^2 x)dx}{(2 \sec x + 5 \tan x)^2}$$

Put $\quad 2 \sec x + 5 \tan x = t$

$\therefore \quad (2 \sec x \tan x + 5 \sec^2 x)\, dx = dt$

$\therefore \quad I = \int \frac{dt}{t^2} = -\frac{1}{t} + c$

$$= -\frac{1}{(2 \sec x + 5 \tan x)} + c$$

$$= -\frac{\cos x}{(2 + 5 \sin x)} + c$$

(II) $\int \frac{(a \cos x + b)}{(a + b \cos x)^2}\, dx.$

Rule : Divide above and below by $\sin^2 x$, then

Put $\quad a \operatorname{cosec} x + b \cot x = t$

Example : Evaluate : $\int \frac{(\cos x + 2)}{(1 + 2 \cos x)^2}\, dx.$

Solution. Let $I = \int \frac{(\cos x + 2)}{(1 + 2 \cos x)^2}\, dx$

Divide above and below by $\sin^2 x$, then

$$I = \int \frac{(\operatorname{cosec} x \cot x + 2 \operatorname{cosec}^2 x)}{(\operatorname{cosec} x + 2 \cot x)^2}$$

Put $\operatorname{cosec} x + 2 \cot x = t$

$\therefore \quad (- \operatorname{cosec} x \cot x - 2 \operatorname{cosec}^2 x)\, dx = dt$

or $\quad (\operatorname{cosec} x \cot x + 2 \operatorname{cosec}^2 x)\, dx = -dt$

then $\quad I = \int \frac{-dt}{t^2} = \frac{1}{t} + c$

$$= \frac{1}{(\operatorname{cosec} x + 2 \cot x)} + c$$

$$= \frac{\sin x}{(1 + 2 \cos x)} + c.$$

6. Integrals of the form :

(i) $\quad \int (\sin x + \cos x)\, f(\sin 2x)dx$
(ii) $\quad \int (\sin x - \cos x)\, f(\sin 2x)dx$

(i) $\int (\sin x + \cos x)\, f(\sin 2x)dx$

Rule : Since integral for sin x + cos x is sin x − cos x

Put sinx − cos x = t ∴ (cos x + sin x) dx = dt

and $\sin^2 x + \cos^2 x - 2 \sin x \cos x = t^2$ or $\sin 2x = 1 - t^2$

then $\int (\sin x + \cos x) f(\sin 2x) dx = \int f(1 - t^2) dt$ it can be evaluated easily.

Examples : Evaluate : $\int (\sqrt{\tan x} + \sqrt{\cot x}) \, dx$.

Solution. Let I = $\int (\sqrt{\tan x} + \sqrt{\cot x}) \, dx$

$$= \int \left(\sqrt{\left(\frac{\sin x}{\cos x} \right)} + \sqrt{\left(\frac{\cos x}{\sin x} \right)} \right) dx$$

$$= \int \frac{(\sin x + \cos x)}{\sqrt{\sin x \cos x}} dx$$

Put sin x − cos x = t ∴ (cos x + sin x) dx = dt

and $\sin^2 x + \cos^2 x - 2 \sin x \cos x = t^2$

⇒ $1 - 2 \sin x \cos x = t^2 \Rightarrow \sin x \cos x = \frac{1 - t^2}{2}$

then I = $\int \dfrac{dt}{\sqrt{\left(\dfrac{1 - t^2}{2} \right)}} = \sqrt{2} \int \dfrac{dt}{\sqrt{(1 - t^2)}}$

$$= \sqrt{2} \, \sin^{-1}(t) + c$$

$$= \sqrt{2} \, \sin^{-1}(\sin x - \cos x) + c$$

(ii) $\int (\sin x - \cos x) f(\sin 2x) dx$

Rule : Since integral of sin x − cos x is − cos x − sin x

Put sin x + cos x = t ∴ (cos x − sin x) dx = dt

and $\sin^2 x + \cos^2 x + 2 \sin x \cos x = t^2$

or $\sin 2x = t^2 - 1$

then $\int (\sin x - \cos x) f(\sin 2x) dx = - \int f(t^2 - 1) dt$ it can be evaluated easily.

Examples : Evaluate : $\int \dfrac{(\sin x - \cos x) dx}{(\sin x + \cos x) \sqrt{(\sin x \cos x + \sin^2 x \cos^2 x)}}$.

Solution. Let I = $\int \dfrac{(\sin x - \cos x) dx}{(\sin x + \cos x) \sqrt{(\sin x \cos x + \sin^2 x \cos^2 x)}}$

Put sin x + cos x = t ∴ (cos x − sin x) dx = dt

⇒ (sin x − cos x) dx = − dt

and $\sin^2 x + \cos^2 x + 2 \sin x \cos x = t^2$

⇒ $\sin x \cos x = \frac{t^2 - 1}{2}$

then $I = \int \dfrac{-dt}{t\sqrt{\left\{ \left(\dfrac{t^2-1}{2} \right) + \left(\dfrac{t^2-1}{2} \right)^2 \right\}}}$

$= -\int \dfrac{dt}{t\sqrt{\left\{ \left(\dfrac{t^2-1}{2} \right) \left(1 + \dfrac{t^2-1}{2} \right) \right\}}}$

$= -\int \dfrac{2\,dt}{t\sqrt{(t^4-1)}}$

$= -2\int \dfrac{t^3\,dt}{t^4\sqrt{(t^4-1)}}$

Again put $t^4 - 1 = z^2$

$\Rightarrow$ $4t^3\ dt = 2z\ dz$ or $2t^3\ dt = z\ dz$

then $I = -\int \dfrac{z\,dz}{(z^2+1)z} = -\int \dfrac{dz}{(z^2+1)} = -\tan^{-1} z + c$

$= -\tan^{-1}\left(\sqrt{(t^4-1)}\right) + c$

$= -\tan^{-1}\left(\sqrt{\{(\sin x + \cos)^4 - 1\}}\right) + c$

7. Integral of the form :

$$\int R\,(\sin x,\, \cos x)\ dx$$

Rule :

(i)   If $R\,(-\sin x,\, \cos x) = -R\,(\sin x,\, \cos x)$ then put $\cos x = t$

(ii)   If $R\,(\sin x,\, -\cos x) = -R\,(\sin x,\, \cos x)$ then put $\sin x = t$

(iii)   If $R\,(-\sin x,\, -\cos x) = R\,(\sin x,\, \cos x)$ then put $\tan x = t$

Examples : Evaluate : $\displaystyle\int \dfrac{dx}{\sin x(2\cos^2 x - 1)}$.

Solution. Let $I = \displaystyle\int \dfrac{dx}{\sin x(2\cos^2 x - 1)}$

Let $R\,(\sin x,\, \cos x) = \dfrac{1}{\sin x(2\cos^2 x - 1)}$

$\Rightarrow$ $R = (-\sin x,\, \cos x) = \dfrac{1}{-\sin x(2\cos^2 x - 1)} = -R\,(\sin x,\, \cos x)$

therefore we put $\cos x = t$ $\therefore$ $\sin x\ dx = -dt$

$\because$ $I = \displaystyle\int \dfrac{dx}{\sin x(2\cos^2 x - 1)} = \int \dfrac{\sin x\ dx}{(1 - \cos^2 x)(2\cos^2 x - 1)}$

$= \displaystyle\int \dfrac{dt}{(1 - t^2)(2t^2 - 1)} = \int \dfrac{dt}{(1 - t^2)(1 - 2t^2)}$

$$= \int \left(\frac{-1}{1-t^2} + \frac{2}{1-2t^2} \right) dt$$

$$= -\int \frac{dt}{(1-t^2)} + 2\int \frac{dt}{(1-2t^2)}$$

$$= -\int \frac{dt}{(1-t^2)} + \frac{2}{2}\int \frac{dt}{\left\{ \left(\frac{1}{\sqrt{2}} \right)^2 - t^2 \right\}}$$

$$= -\frac{1}{2} \ln\left| \frac{1+t}{1-t} \right| + \frac{1}{2 \cdot (1/\sqrt{2})} \ln\left| \frac{1/\sqrt{2}+t}{1/\sqrt{2}-t} \right| + c$$

$$= -\frac{1}{2} \ln\left| \frac{1+t}{1-t} \right| + \frac{1}{\sqrt{2}} \ln\left| \frac{1+t\sqrt{2}}{1-t\sqrt{2}} \right| + c$$

$$= -\frac{1}{2} \ln\left| \frac{1+\cos x}{1-\cos x} \right| + \frac{1}{\sqrt{2}} \ln\left| \frac{1+\sqrt{2}\cos x}{1-\sqrt{2}\cos x} \right| + c$$

8. Integrals of the form :

(i) $\quad \int \frac{dx}{(a+b\sin x)}$

(ii) $\quad \int \frac{dx}{(a+b\cos x)}$

(iii) $\quad \int \frac{dx}{(a\sin x + b\cos x)}$

(iv) $\quad \int \frac{dx}{(a\sin x + b\cos x + c)}$

(v) $\quad \int \frac{\phi(\tan x / 2)dx}{(a\sin x + b\cos x + c)}$

(vi) $\quad \int \frac{(p\cos x + q\sin x)}{(a\cos x + b\sin x)} dx$

(vii) $\quad \int \frac{(p\cos x + q\sin x + r)}{(a\cos x + b\sin x + c)} dx$

Rule for (i), (ii), (iii), (iv), (v) :

Write $\cos x = \dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}, \sin x = \dfrac{2\tan(x/2)}{1+\tan^2(x/2)}$

the numerator will become $\sec^2(x/2)$ and the denominator will be a quadratic in $\tan(x/2)$. Putting $\tan(x/2) = t$ i.e., $\sec^2(x/2) dx = 2 dt$ the question will reduces to the form

$$\int \frac{dt}{at^2 + bt + c} \qquad \text{or} \qquad \int \frac{\phi(t)dt}{at^2 + bt + c}$$

Examples : Evaluate : $\int \dfrac{dx}{(5 + 4\cos x)}$.

Solution. Let $I = \int \dfrac{dx}{(5 + 4\cos x)} = \int \dfrac{dx}{5 + 4\left(\dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)} \right)}$

$$= \int \frac{\sec^2(x/2)\, dx}{(9 + \tan^2(x/2))}$$

Put $\quad \tan(x/2) = t \qquad\qquad\qquad \therefore \qquad \sec^2(x/2)\, dx = 2\, dt$

then $\quad I = 2 \int \dfrac{dt}{3^2 + t^2} = \dfrac{2}{3} \tan^{-1} \left(\dfrac{t}{3} \right) + c$

$\qquad\qquad = \dfrac{2}{3} \tan^{-1} \left(\dfrac{\tan(x/2)}{3} \right) + c$

9. Integrals of the form :

(i) $\quad \displaystyle\int \dfrac{dx}{(a + b\sin x)^2}$ $\qquad\qquad$ (ii) $\quad \displaystyle\int \dfrac{dx}{(a + b\cos x)^2}$

(I) $\quad \displaystyle\int \dfrac{dx}{(a + b\sin x)^2}$

Rule : Put $t = \dfrac{(a\sin x + b)}{(a + b\sin x)}$ $\qquad\qquad\qquad\qquad$ ...(1)

$\therefore \quad dt = \dfrac{(a^2 - b^2)\cos x}{(a + b\sin x)^2} dx \qquad$ or $\qquad \dfrac{dt}{(a^2 - b^2)\cos x} = \dfrac{dx}{(a + b\sin x)^2}$ $\qquad$ ...(2)

Example : Evaluate $\displaystyle\int \dfrac{dx}{\left(5 + 4\sin x\right)^2}$

put $\quad t = \dfrac{5\sin x + 4}{5 + 4\sin x}$ $\qquad\qquad\qquad\qquad$ ...(1)

$\therefore \quad dt = \dfrac{(5^2 - 4^2)\cos x}{(5 + 4\sin x)^2} dx$ $\qquad\qquad\qquad$ ...(2)

From (1), $\sin x = \dfrac{5t - 4}{5 - 4t}$ then $\cos x = \sqrt{1 - \sin^2 x} = \dfrac{3\sqrt{(1 - t^2)}}{(5 - 4t)}$

From (2), $\dfrac{(5 - 4t)dt}{27\sqrt{(1 - t^2)}} = \dfrac{dx}{(5 + 4\sin x)^2}$

Integrating both sides, we get

$$\int \dfrac{dx}{(5 + 4\sin x)^2} = \dfrac{5}{27} \int \dfrac{dt}{\sqrt{(1 - t^2)}} + \dfrac{4}{27} \int \dfrac{-t\, dt}{\sqrt{(1 - t^2)}}$$

Let $1 - t^2 = z^2 \Rightarrow -t\, dt = z\, dz$

$\qquad = \dfrac{5}{27} \sin^{-1} t + \dfrac{4}{27} \int \dfrac{z\, dz}{z} + c$

$\qquad = \dfrac{5}{27} \sin^{-1} t + \dfrac{4}{27} \int dz + c$

$\qquad = \dfrac{5}{27} \sin^{-1} t + \dfrac{4}{27} z + c$

$\qquad = \dfrac{5}{27} \sin^{-1} t + \dfrac{4}{27} \sqrt{(1 - t^2)} + c$

$$= \frac{5}{27}\sin^{-1}\left\{\frac{5\sin x + 4}{5 + 4\sin x}\right\} + \frac{4}{9}\frac{\cos x}{(5 + 4\sin x)} + c \qquad \left\{\because t = \frac{5\sin x + 4}{5 + 4\sin x}\right\}$$

(ii) $\int \dfrac{dx}{(a + b\cos x)^2}.$

Rule : Put $t = \dfrac{(a\cos x + b)}{(a + b\cos x)}$ ...(1)

$\therefore \quad dt = \dfrac{(b^2 - a^2)\sin x}{(a + b\cos x)^2}dx \qquad$ or $\qquad \dfrac{dt}{(b^2 - a^2)\sin x} = \dfrac{dx}{(a + b\cos x)^2}$

Now substitute the value of sin x in L.H.S. from (1) in terms of t and then integrate both sides.

Examples : Evaluate : $\int \dfrac{dx}{(12 + 13\cos x)^2}.$

Solution. Let $I = \int \dfrac{dx}{(12 + 13\cos x)^2}$

Put $\quad t = \dfrac{(12\cos x + 13)}{(12 + 13\cos x)}$ ...(1)

$\therefore \quad dt = \dfrac{25\sin x}{(12 + 13\cos x)^2}dx \qquad$ or $\qquad \dfrac{dt}{25\sin x} = \dfrac{dx}{(12 + 13\cos x)^2}$ ...(2)

From (1), $\cos x = \dfrac{12t - 13}{12 - 13t} \qquad \therefore \quad \sin x = \dfrac{5\sqrt{(t^2 - 1)}}{(12 - 13t)}$

From (2), $\dfrac{(12 - 13t)dt}{125\sqrt{(t^2 - 1)}} = \dfrac{dx}{(12 + 13\cos x)^2}$

Integrating both sides, we get

$$\int \frac{dx}{(12 + 13\cos x)^2} = \frac{12}{125}\int \frac{dt}{\sqrt{(t^2 - 1)}} - \frac{13}{125}\int \frac{t\,dt}{\sqrt{(t^2 - 1)}}$$

$$= \frac{12}{125}\ln|t + \sqrt{(t^2 - 1)}| - \frac{13}{125}\sqrt{(t^2 - 1)} + c$$

$$= \frac{12}{125}\ln\left|\frac{5\sin x + 12\cos x + 13}{12 + 13\cos x}\right| - \frac{13}{25}\cdot\frac{\sin x}{(12 + 13\cos x)} + c \qquad \left\{\because t = \frac{12\cos x + 13}{12 + 13\cos x}\right\}$$

10. Integrals of the form :

(i) $\quad \int \sqrt{(\sec^2 x \pm a)}\ dx$

(ii) $\quad \int \sqrt{(\csc^2 x \pm a)}\ dx$

(ii) $\quad \int \sqrt{(\tan^2 x \pm a)}\ dx$

(iv) $\quad \int \sqrt{(\cot^2 x \pm a)}\ dx$

(I) $\int \sqrt{(\sec^2 x \pm a)} \, dx$

Rule :

Let $\quad$ I = $\sqrt{(\sec^2 x \pm a)} \, dx = \int \dfrac{(\sec^2 x \pm a)}{\sqrt{(\sec^2 x \pm a)}} \, dx$

$$= \int \dfrac{\sec^2 x \, dx}{\sqrt{(\sec^2 x \pm a)}} \pm a \int \dfrac{dx}{\sqrt{(\sec^2 x \pm a)}}$$

$$= \int \dfrac{\sec^2 x \, dx}{\sqrt{(1 \pm a) + \tan^2 x}} \pm a \int \dfrac{\cos x \, dx}{\sqrt{1 \pm a(1 - \sin^2 x)}}$$

In first integral on R.H.S. put tan x = t and in second integral put sin x = u

Example : Evaluate : $\int \sqrt{(\sec^2 x - 2)} \, dx$

Solution. Let I = $\int \sqrt{(\sec^2 x - 2)} \, dx$

$$= \int \dfrac{(\sec^2 x - 2)}{\sqrt{(\sec^2 x - 2)}} dx$$

$$= \int \dfrac{\sec^2 x \, dx}{\sqrt{(\sec^2 x - 2)}} - 2 \int \dfrac{dx}{\sqrt{(\sec^2 x - 2)}}$$

$$= \int \dfrac{\sec^2 x \, dx}{\sqrt{(\tan^2 x - 1)}} - 2 \int \dfrac{\cos x \, dx}{\sqrt{1 - 2(1 - \sin^2 x)}}$$

$$= \int \dfrac{\sec^2 x \, dx}{\sqrt{(\tan^2 x - 1)}} - 2 \int \dfrac{\cos x \, dx}{\sqrt{(\sqrt{2} \sin x)^2 - 1}}$$

In first integral on R.H.S. put tan x = t and in second integral put $\sqrt{2}$ sin x = u.

i.e., sec² x dx = dt and cos x dx = $\dfrac{du}{\sqrt{2}}$

$\therefore \quad$ I = $\int \dfrac{dt}{\sqrt{(t^2 - 1)}} - \dfrac{2}{\sqrt{2}} \int \dfrac{du}{\sqrt{(u^2 - 1)}}$

$$= - \ln (t + \sqrt{(t^2 - 1)}) - \sqrt{2} \ln \left(u + \sqrt{u^2 - 1} \right) + c$$

$$= - \ln (\tan x + \sqrt{(\tan^2 x - 1)}) - \sqrt{2} \ln \left(\sqrt{2} \sin x + \sqrt{2 \sin^2 x - 1} \right) + c$$

(iii) $\int \sqrt{(\tan^2 x \pm a)} dx$ and

(iv) $\sqrt{(\cot^2 x \pm a)} \, dx$

Rule : In this case change $\tan^2 x$ into $\sec^2 x - 1$ and $\cot^2 x$ into $\mathrm{cosec}^2 x - 1$ and then proceed as in (i) and (ii).

(vi) Integrals of the form

(i) $\int f(e^{ax})dx$

(ii) $\int\left(\dfrac{ae^{ax} + be^{-x}}{pe^{x} + qe^{-x}}\right)dx$

(i) $\int f(e^{ax})dx$

Rule : Transform into an integral of a rational function by the substitution $e^{ax} = t$.

Example : Evaluate : $\int\dfrac{dx}{(1+e^{x})}$

Solution. Let $I = \int\dfrac{dx}{(1+e^{x})}$

Divide above and below by e^x then

$$I = \int\dfrac{e^{-x}}{1+e^{-x}}dx$$

Put $\quad 1 + e^{-x} = t \qquad\qquad \therefore \quad e^{-x}\,dx = -dt$

then $\quad I = -\int\dfrac{dt}{t}$

$\qquad = -\ln t + c$

$\qquad = -\ln(1 + e^{-x}) + c$

(ii) $\int\left(\dfrac{ae^{x} + be^{-x}}{pe^{x} + qe^{-x}}\right)dx$

Rule : Express numerator as l (Denominator) $+ m\dfrac{d}{dx}$ (Denominator)

Find l and m by comparing the coefficients of e^x and e^{-x} and split the integral into sum of two integrals as

$$l\int dx + m\int\dfrac{dc.\ of\ (Denominator)}{(Denominator)}dx$$

$$= lx + m\ln\ |\ Denominator\ | + c$$

Example : Evaluate : $\int\left(\dfrac{4e^{x} + 6e^{-x}}{9e^{x} - 4e^{-x}}\right)dx$

Solution. Let $I = \int\left(\dfrac{4e^{x} + 6e^{-x}}{9e^{x} - 4e^{-x}}\right)dx$

Express Numerator as

$\Rightarrow\qquad$ Numerator $= l$(Denominator) $+ m\dfrac{d}{dx}$(Denominator)

or $\qquad (4e^{x} + 6e^{-x}) = l(9e^{x} - 4e^{-x}) + m(9e^{x} + 4e^{-x})$

Comparing the coefficients of e^x and e^{-x} on both sides then

$\qquad 4 = 9l + 9m$ and $6 = -4l + 4m$

After solving we get

$$l = -\frac{19}{36} \quad \text{and} \quad m = \frac{35}{36}$$

then $\quad I = \int \dfrac{-\dfrac{19}{36}(9e^x - 4e^{-x}) + \dfrac{35}{36}(9e^x + 4e^{-x})}{(9e^x - 4e^{-x})} dx$

$$= -\frac{19}{36}\int dx + \frac{35}{36}\int \left(\frac{9e^x + 4e^{-x}}{9e^x - 4e^{-x}}\right) dx$$

$$= -\frac{19}{36}x + \frac{35}{36} \ln (9e^x - 4e^{-x}) + c.$$

(viii) The Integral of A Binomial Differential

A binomial differential is a differential of the form $x^m (a + bx^n)^p dx$ where m, n, p $\in$ Q and a, b are constants not equal to zero. The integral

$$\int x^m (a + bx^n)^p dx$$

is expressible in terms of elementary functions in the following three cases :

Case 1 : (i) If $p \in N$ then expand by the formula of Newton Binomial.

(ii) If p < 0, then we put $x = t^k$ where k is the common denominator of the fractions m and n.

Case 2 : If $\dfrac{m+1}{n}$ = Integer then we put $a + bx^n = t^\alpha$ where α is the denominator of the fraction p.

Example : Evaluate : $\int (x^2 + x)(x^{-8} + 2x^{-9})^{1/10} dx$

Solution. Let $I = \int (x^2 + x)(x^{-8} + 2x^{-9})^{1/10} dx$

$$= \int (x + 1).x.(x^{-8} + 2x^{-9})^{1/10} dx$$

$$= \int (x + 1)(x^2 + 2x)^{1/10} dx$$

Put $\quad x^2 + 2x = t$

$\therefore \qquad 2(x + 1)dx = dt \quad \Rightarrow \quad (x + 1) dx = \dfrac{dt}{2}$

$\therefore \qquad I = \dfrac{1}{2}\int t^{1/10} dt = \dfrac{1}{2}.\dfrac{t^{1/10 + 1}}{\left(\dfrac{1}{10} + 1\right)} = \dfrac{5}{11} t^{11/10} + c$

$$= \frac{5}{11}(x^2 + 2x)^{11/10} + c$$

2. DEFINITE INTEGRALS AND THEIR PROPERTIES

Definition : Let f be a function which is continuous on the closed interval [a, b]. The definite integral of f(x) from a to b is defined to be the limit

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\sum_{i=1}^{n} f(x_i)\Delta x$

is a Riemann sum of f(x) on [a, b]

Newton-Leibnitz Formula

Let f be a function of x defined in the closed interval [a, b] and the function F is an antiderivative of f on [a, b], such that F'(x) = f(x) for all x in the domain of f, then

$$\int_a^b f(x)\,dx = F(x) + c \Big|_{x=a}^{x=b}$$

$$= (F(b) + c) - (F(a) + c)$$

$$= F(b) - F(b)$$

Note : In definite integrals, constant of integration is never present.

Rule for Finding Definite Integral

To evaluate the definite integral $\int_a^b f(x)\,dx$ (a < b)

First find out the indefinite integral of f(x) i.e., $\int f(x)dx$, leaving the constant of integration c.

Let $\quad \int f(x)\,dx = F(x)$ then $\int_a^b f(x)\,dx = F(x)\Big|_{x=a}^{x=b}$

$$= \lim_{x \to b-} F(x) - \lim_{x \to a+} F(x)$$

To evaluate above the following cases may arise :

Case I : If f(x) is continuous at x = a and x = b :

$\therefore \quad \lim_{x \to a+} F(x) = F(a) \qquad$ and $\qquad \lim_{x \to b-} F(x) = F(b)$

then $\quad \int_a^b f(x)\,dx = F(b) - F(a)$

Case II : If f(x) is continuous at x = a and discontinuous at x = b :

$\therefore \quad \lim_{x \to a+} F(x) = F(a)$

then $\quad \int_a^b f(x)\,dx = \lim_{x \to b-} F(x) - F(a)$

Case III : If f(x) is discontinuous at x = a and continuous at x = b :

$\therefore \quad \lim_{x \to b-} F(x) = F(b)$

then $\quad \int_a^b f(x)\,dx = F(b) - \lim_{x \to a+} F(x)$

Case IV : If f(x) is discontinuous at x = c (a < c < b) :

∵ f(x) is discontinuous at x = c or f(x) has different values in (a, c) and (c, b) then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Note : If f(x) is not defined at x = a and x = b and defined in open interval (a, b) then $\int_a^b f(x)\,dx$ can be evaluated.

Example : Evaluate : $\int_{-1}^3 x^4\,dx$.

Solution. $\int_{-1}^3 x^4\,dx = \dfrac{x^5}{5}\Big|_{-1}^3 = \dfrac{1}{5}(3^5 - (-1)^5)$

$$= \frac{244}{5}$$

Geometrical Interpretation of the Definite Integral

First we construct the graph of the integrand y = f(x) then in the case of f(x) ≥ 0 ∀ x ∈ [a, b], the integral $\int_a^b f(x)\,dx$ is numerically equal to the area bounded by the curve y = f(x), the x-axis and the ordinates x = a and x = b.
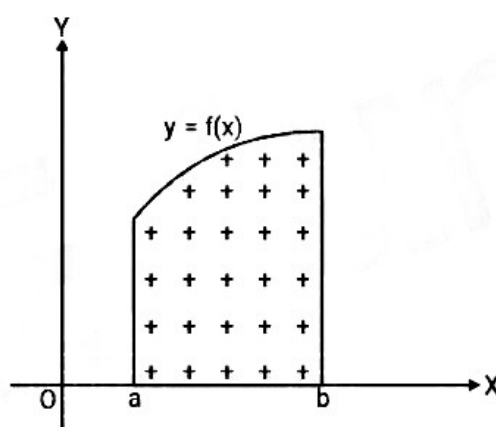
Fig.

OR

$\int_a^b f(x)\,dx$ is numerically equal to the area of curvilinear trapezoid bounded by the given curve, the straight lines x = a and x = b, and the x-axis.

In general : $\int_a^b f(x)dx$ represents an algebraic sum of areas of the region bounded by the curve y = f(x), the x-axis and the ordinates x = a and
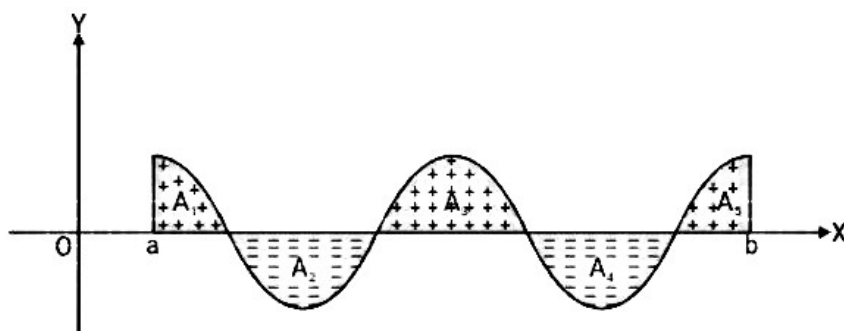
Fig.

x = b. The areas above the x-axis enter into this sum with a positive sign, while those below the x-axis enter it with a negative sign.

$$\therefore \qquad \int_a^b f(x)\,dx = A_1 - A_2 + A_3 - A_4 + A_5$$

where A_1, A_2, A_3, A_4, A_5 are the areas of the shaded regions.

Example : Evaluate : $\int_{-1}^{1} \max\{2-x, 2, 2+x\}dx$.

Solution. To find $\int_{-1}^{1} \max\{2-x, 2, 1+x\}$

when $\quad x < 0$

$\qquad y = \max\{2 - x, 2, 1 + x\} = 2 - x$

when $\quad x > 0$

$\qquad y = \max\{2 - x, 2, 1 + x\} = 2$

Plotting these curves on the XY plane
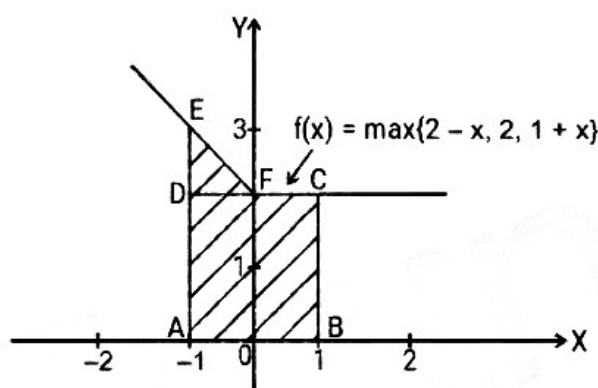
$\int_{-1}^{1} \max\{2-x, 2, 1+x\}$ = Area of shaded region

$\qquad$ = Area of square ABCD + Area of $\triangle$DEF

$\qquad = 2 \times 2 + \dfrac{1}{2} \times 1 \times 1$

$\qquad = 4 + \dfrac{1}{2}$

$\qquad = \dfrac{9}{2}$

Example : Evaluate : $\int_{-1}^{1} \min\{|x|, |x-1|, |x+1|\}\,dx$

Solution. The graph of $f(x) = \min\{|x|, |x-1|, |x+1|\}$ is shown as in Fig..
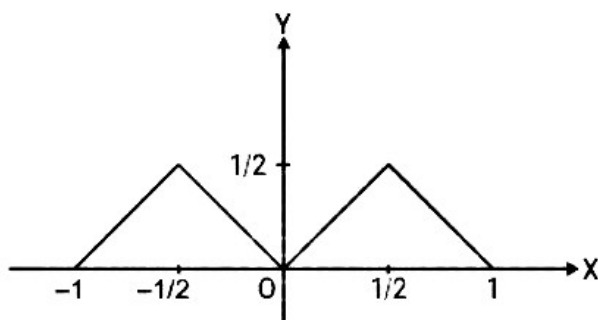
Fig.

Therefore $\int_{-1}^{1} \min\{|x|, |x-1|, |x+1|\} \, dx$ = Area of shaded region

$$= 2 \times \left(\frac{1}{2} \times 1 \times \frac{1}{2}\right) = \frac{1}{2}.$$

Example : Evaluate : $\int_{-2}^{2} \max\{x + |x|, x - [x]\} \, dx$, where [x] denotes the greatest integer $\leq$ x.

Solution. $\quad I = \int_{-2}^{2} \max\{x + |x|, x - [x]\} \, dx.$

$$x + |x| = \begin{cases} x - x = 0, & -2 < x < -1 \\ x - x = 0, & -1 < x < 0 \\ x + x = 2x, & 0 < x < 1 \\ x + x = 2x, & 1 < x < 2 \end{cases}$$

$$= \begin{cases} 0, & -2 < x < -1 \\ 0, & -1 < x < 0 \\ 2x, & 0 < x < 1 \\ 2x, & 1 < x < 2 \end{cases}$$

$$x - [x] = \begin{cases} x + 2 & -2 < x < -1 \\ x + 1 & -1 < x < 0 \\ x & 0 < x < 1 \\ x - 1 & 1 < x < 2 \end{cases}$$

$I = \int_{-2}^{-1} \max\{0, x + 2\} \, dx + \int_{-1}^{0} \max\{0, x + 1\} \, dx + \int_{0}^{1} \max\{2x, x\} \, dx + \int_{1}^{2} \max\{2x, x + 1\} \, dx$

$I = \int_{-2}^{-1}(x + 2) \, dx + \int_{-1}^{0}(x + 1) \, dx + \int_{0}^{1} 2x \, dx + \int_{1}^{2} 2x \, dx$

$I = \left[\frac{x^2}{2} + 2x\right]_{-2}^{-1} + \left[\frac{x^2}{2} + x\right]_{-1}^{0} + \left[x^2\right]_{0}^{1} + \left[x^2\right]_{1}^{2}$

$I = \left[\frac{1}{2} - 2 - 2 + 4\right] + \left[0 + 0 - \frac{1}{2} + 1\right] + [1 - 0] + [4 - 1]$

$I = \frac{1}{2} + \frac{1}{2} + 1 + 3$

$= 1 + 1 + 3$

$= 5$

Definite Integral by standard methods of indefinite integral

(i) **Transformation Method :** To evaluate the integral $\int_{a}^{b} f(x) \, dx$ we can transform f(x) into other function $\phi(x)$ without any substitution then

$$\int_{a}^{b} f(x) \, dx = \int_{a}^{b} \phi(x) \, dx$$

and then $\int_{a}^{b} \phi(x) \, dx$ evaluate easily.

Example : Evaluate : $\int_{0}^{\pi} \sqrt{(1 + \cos 2x)} \, dx$

Solution. Let $I = \int_0^\pi \sqrt{(1 + \cos 2x)}\, dx$

$$= \int_0^\pi \sqrt{2}\, |\cos x|\, dx \qquad\qquad (\text{ since } \cos 2x = 2\cos^2 x - 1\,)$$

$$= \sqrt{2} \int_0^{\pi/2} |\cos x|\, dx + \sqrt{2} \int_{\pi/2}^\pi |\cos x|\, dx$$

$$= \sqrt{2} \int_0^{\pi/2} \cos x\, dx - \sqrt{2} \int_{\pi/2}^\pi \cos x\, dx$$

$$= \sqrt{2}\, (\sin x)\Big|_0^{\pi/2} - \sqrt{2}\, (\sin x)\Big|_{\pi/2}^\pi$$

$$= \sqrt{2}\, (1 - 0) - \sqrt{2}(0 - 1) = 2\sqrt{2}$$

(ii) Substitution Method : By the substitution $x = \phi(x)$ in the definite integral $\int_a^b f(x)\, dx$, the lower and the upper limits change.

When $\qquad x = a$ then $a = \phi(t) \Rightarrow t = \phi^{-1}\,(a)$

and when $\qquad x = b$ then $b = \phi(t) \Rightarrow t = \phi^{-1}\,(b)$

Hence the new lower and upper limits are $\phi^{-1}\,(a)$ and $\phi^{-1}\,(b)$ respectively.

Again put $\ln\,(t + \sqrt{(1 + t^2)}) = z$

$$\therefore \qquad t + \sqrt{(1 + t^2)} = e^z \qquad\qquad \Rightarrow \qquad \sqrt{(1 + t^2)} = (e^z - t)$$

$$\Rightarrow \qquad 1 + t^2 = e^{2z} + t^2 - 2te^z \qquad \text{or} \qquad t = \left(\frac{e^z - e^{-z}}{2}\right)$$

$$\therefore \qquad dt = \left(\frac{e^z + e^{-z}}{2}\right) dz$$

When $t = 0 \Rightarrow z = 0$

$\qquad t = \infty \Rightarrow z = \infty$

$$\therefore \qquad I = \int_0^\infty \frac{(e^z + e^{-z})}{2(e^z)^n}\, dz$$

$$= \frac{1}{2} \int_0^\infty (e^{(1-n)z} + e^{-(n+1)z})\, dz$$

$$= \frac{1}{2} \left\{ (0 - 0) - \left(\frac{1}{1-n} - \frac{1}{n+1}\right) \right\}$$

$$= \frac{1}{2} \left\{ \frac{1}{1-n} - \frac{1}{n+1} \right\}$$

$$= \frac{n}{1 - n^2}$$

(iii) By Parts Method : Let u and v are the functions of x, u and v have continuous derivatives on the interval [a, b]. Then

$$\int_a^b uv' dx = uv \Big|_a^b - \int_a^b u'v \; dx$$

where u' and v' are the derivatives of u and v respectively.

Example : Evaluate : $\int_{\pi/4}^{\pi/3} x \; \cosec^2 x \; dx$.

Solution. Integration by parts, Definite Integral

$$\boxed{\int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du}$$

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} x \cdot \cosec^2 x \; dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} x \cdot d(-\cot x)$$

$$= x\left[(-\cot x)\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (-\cot x).dx$$

$$= x\left[-\cot\frac{\pi}{3} + \cot\frac{\pi}{4}\right] + \left[\log \sin\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= x\left[-\frac{1}{\sqrt{3}} + 1\right] + \left(\log\sin\left(\frac{\pi}{3}\right) - \log\sin\frac{\pi}{4}\right)$$

$$= x\left(1 - \frac{1}{\sqrt{3}}\right) + \left[\log\left(\frac{\sqrt{3}}{2}\right) - \log\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$= x\left(1 - \frac{1}{\sqrt{3}}\right) + \log\left(\frac{\sqrt{3}/2}{1/\sqrt{2}}\right)$$

$$= x\left(1 - \frac{1}{\sqrt{3}}\right) + \log\left(\frac{\sqrt{3}}{\sqrt{2}}\right)$$

$$= x\left(1 - \frac{1}{\sqrt{3}}\right) + \log\left(\frac{3}{2}\right)^{1/2}$$

$$= \frac{\sqrt{3}\left(\sqrt{3} - 1\right)}{3} x + \frac{1}{2}\log\frac{3}{2}$$

Properties of The Definite Integral

Some simple properties of definite integrals can be derived from the basic definition, or from the Fundamental Theorem of the Calculus.

(a) $\int_a^a f(x) \; dx = 0$

If the upper and lower limits of the integral are the same, the integral is zero. This becomes obvious if we have a positive function and can interpret the integral in terms of 'the area under a curve'.

(b)   If $a \leq b \leq c$,

$\int_a^c f(x) \; dx = \int_a^b f(x) \; dx + \int_b^c f(x) \; dx$
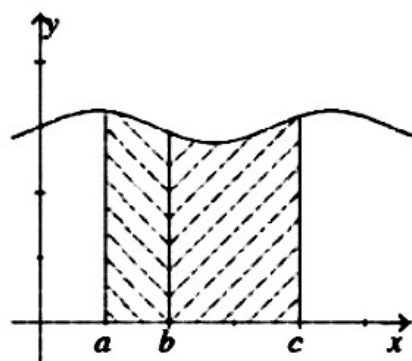
This says that the integral of a function over the union of two intervals is equal to the sum of the integrals over each of the intervals. The diagram opposite helps to make this clear if f(x) is a positive function.

(c) $\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx$ for any constant c.

This tells us that we can move a constant past the integral sign but we can only do this with constants, never with variables!

(d) $\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$

That is, the integral of sum is equal to the sum of the integrals.

(e)   If f(x) ≤ g(x) in [a, b] then

$\int_a^b f(x)\, dx \le \int_a^b g(x)\, dx$.
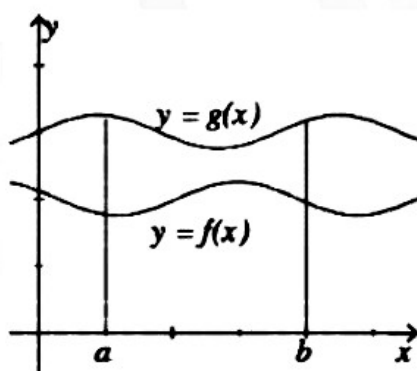
That is, integration preserves inequalities between functions. The diagram opposite explains this result if f(x) and g(x) are positive functions.

(f) $\int_a^b c\, dx = c(b - a)$
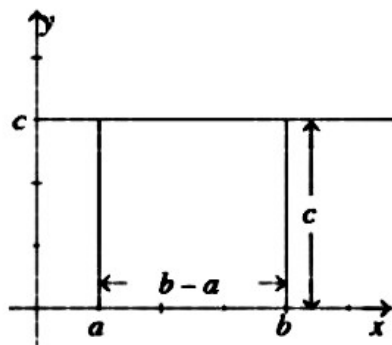
This tells us that the integral of a constant is equal to the product of the constant and the range of integration. It becomes obvious when we look at the diagram with c > 0, since the area represented by the integral is just a rectangle of height c and width b – a.

(g)   We can combine (e) and (f) to give the result that, if M is any upper bound and m any lower bound for f(x) in the interval [a, b], so that m ≤ f(x) ≤ M, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$
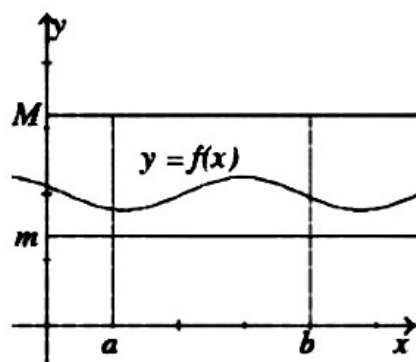
This, too becomes clear when f(x) is a positive function and we can interpret the integral as the area under the curve.

(h)   Finally we extend the definition of the definite integral slightly, to remove the restriction that the lower limit of the integral must be a smaller number than the upper limit. We do this by specifying that

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

For example,

$$\int_2^1 f(x)\,dx = -\int_1^2 f(x)\,dx.$$

Example : Evaluate : $\int_a^b \dfrac{|x|}{x}\,dx$, a < b

Solution. Let I = $\int_a^b \dfrac{|x|}{x}\,dx$

Case I : When 0 < a < b

then   I = $\int_a^b \dfrac{x}{x}\,dx = \int_a^b 1.dx = x\Big|_a^b = b - a$

= | b | − | a | ...(1)

Case II : When a < 0 < b

then   I = $-\int_a^0 1.dx + \int_0^b 1.dx$

= − (0 − a) + (b − 0)

= b + a   ...(2)

= | b | − | a |

Case III : When a < b < 0

then   I = $\int_a^b (-1).dx = -(b-a) = -b + a$

= | b | − | a | ...(3)

It is clear from (1), (2) and (3) we get

$$\int_a^b \frac{|x|}{x}\,dx = |b| - |a|$$

Example : Evaluate : $\int_{-1}^{3/2} |x \sin \pi x| \, dx$

Solution. Let $I = \int_{-1}^{3/2} |x \sin \pi x| \, dx$

$$= \int_{-1}^{1} x \sin \pi x \, dx + \int_{1}^{3/2} (-x \sin \pi x) dx \qquad \left\{ \because |x \sin \pi x| = \begin{cases} x \sin \pi x, & -1 \le x \le 1 \\ -x \sin \pi x, & 1 \le x \le 3/2 \end{cases} \right\}$$

$$= \left\{ -\frac{x \cos \pi x}{x} + \frac{\sin \pi x}{\pi^2} \right\}_{-1}^{1} - \left\{ -\frac{x \cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2} \right\}_{1}^{3/2}$$

$$= \left\{ \left(\frac{1}{\pi} + 0 \right) - \left(-\frac{1}{\pi} + 0 \right) \right\} - \left\{ \left(0 - \frac{1}{\pi^2} \right) \left(\frac{1}{\pi} + 0 \right) \right\}$$

$$= \left(\frac{3}{\pi} + \frac{1}{\pi^2} \right)$$

Some other properties

Property : $\int_0^a f(x) \, dx = \int_0^a f(a - x) \, dx$

Proof : Analytical Method :

$$\text{R.H.S.} = \int_0^a f(a - x) \, dx$$

Put $\quad a - x = t \qquad\qquad\qquad \therefore \quad dx = -dt$

When $x = 0 \Rightarrow t = a$

$\qquad\quad x = a \Rightarrow t = 0$

$\therefore \qquad \text{R.H.S.} = \int_a^0 f(t) (-dt)$

$$= -\int_a^0 f(t) \, dt$$

$$= \int_0^a f(t) \, dt$$

$$= \int_0^a f(x) \, dx$$

$$= \text{L.H.S.}$$

Removal of x : Let $I = \int_0^a x f(x) \, dx$, where $f(x)$ is a function of x whose integral is known and $f(a - x) = f(x)$ then apply above property.

Example : Evaluate : $\int_0^{\pi/2} \dfrac{dx}{1 + \tan^n x}$, $n \in R$

Solution. Let $I = \int_0^{\pi/2} \dfrac{dx}{(1 + \tan^n x)}$

$$I = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x} \, dx \qquad\qquad\qquad \dots(1)$$

$$= \int_0^{\pi/2} \frac{\cos^n\left(\frac{\pi}{2}-x\right)}{\cos^n\left(\frac{\pi}{2}-x\right) + \sin^n\left(\frac{\pi}{2}-x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx \qquad \qquad \text{...(2)}$$

Adding (1) and (2),

$$2I = \int_0^{\pi/2} 1 \cdot dx = \frac{\pi}{2}$$

Hence $I = \pi/4$

Hence remember that :

(i) $\qquad \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} dx$

(ii) $\qquad \int_0^{\pi/2} \frac{dx}{1 + \tan x} = \frac{\pi}{4} = \int_0^{\pi/2} \frac{dx}{1 + \cot x}$

(iii) $\qquad \int_0^{\pi/2} \frac{dx}{1 + \sqrt{(\tan x)}} = \frac{\pi}{4} = \int_0^{\pi/2} \frac{dx}{1 + \sqrt{(\cot x)}}$

(iv) $\qquad \int_0^{\pi/2} \frac{\tan x}{1 + \tan x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cot x}{1 + \cot x} dx$

(v) $\qquad \int_0^{\pi/2} \frac{\sqrt{(\tan x)}}{1 + \sqrt{(\tan x)}} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\sqrt{(\cot x)}}{1 + \sqrt{(\cot x)}} dx$

(vi) $\qquad \int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}} = \frac{\pi}{4} = \int_0^{\pi/2} \frac{dx}{1 + (\cot x)^{\sqrt{2}}}$

Example : Evaluate : $\int_0^1 x(1-x)^{99} dx$

Solution. Let $I = \int_0^1 x(1-x)^{99} dx$

put $\quad t = 1 - x \implies dt = -dx$

at $\quad x = 0 \implies t = 1$

and $\quad x = 1 \implies t = 0$

$$I = -\int_1^0 (1-t)(t)^{99} \, dt$$

$$= \int_0^1 (1-t) \, t^{99} \, dt$$

$$= \int_0^1 (t^{99} - t^{100}) \, dt$$

$$= \left(\frac{t^{100}}{100} - \frac{x^{101}}{101} \right)_0^1$$

$$= \frac{1}{100} - \frac{1}{101}$$

$$= \frac{1}{10100}$$

Example : Evaluate : $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$

Solution. Let I = $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$ ...(1)

$$= \int_0^\pi \frac{(\pi - x) \sin(\pi - x)}{1 + \cos^2(\pi - x)}$$

$$= \int_0^\pi \frac{(\pi - x) \sin x}{1 + \cos^2 x}$$ ...(2)

Adding (1) and (2), we get 2I = $\pi \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx$

Put $\cos x = t$ $\qquad\qquad$ $\therefore$ $\sin x \, dx = -dt$

When $x = 0 \Rightarrow t = 1$

$\qquad\qquad x = \pi \Rightarrow t = -1$

$\therefore \qquad 2I = -\pi \int_1^{-1} \frac{dt}{1 + t^2}$

$$= \pi \int_{-1}^1 \frac{dt}{1 + t^2}$$

$$= \pi \tan^{-1} t \Big|_{-1}^1$$

$$= \pi \{\tan^{-1} 1 - \tan^{-1}(-1)\}$$

$$= \pi \{2 \tan^{-1} 1\}$$

$\therefore \qquad I = \pi \tan^{-1} 1 = \pi \cdot \frac{\pi}{4}$

Hence I = $\frac{\pi^2}{4}$.

Remember that :

(i) $\qquad \int_0^{\pi/2} f(\sin x) dx = \int_0^{\pi/2} f(\cos x) \, dx$

(ii) $\qquad \int_0^{\pi/2} f(\tan x) dx = \int_0^{\pi/2} f(\cot x) \, dx$

(iii) $\qquad \int_0^{\pi/2} f(\sec x) dx = \int_0^{\pi/2} f(\csc x) \, dx$

Property : $\int_a^b f(x) \, dx = \int_a^b f(a + b - x) \, dx$

Note : In 95% cases $a + b = \frac{\pi}{4}, \frac{\pi}{2}, \pi$ and in 5% cases $a + b = 1, 2, 3, ..., \lambda$ etc.

Example : Evaluate : $\int_{\pi/6}^{\pi/3} \frac{1}{1 + \tan^n x} dx, n \in R$

Solution. Let $I = \int_{\pi/6}^{\pi/3} \dfrac{1}{(1 + \tan^n x)}$

$$= \int_{\pi/6}^{\pi/3} \dfrac{\cos^n x}{\cos^n x + \sin^n x}\, dx \qquad \qquad \text{...(1)}$$

$$= \int_{\pi/6}^{\pi/3} \dfrac{\cos^n\left(\dfrac{\pi}{3} + \dfrac{\pi}{6} - x\right)}{\cos^n\left(\dfrac{\pi}{3} + \dfrac{\pi}{6} - x\right) + \sin^n\left(\dfrac{\pi}{3} + \dfrac{\pi}{6} - x\right)}\, dx$$

$$= \int_{\pi/6}^{\pi/3} \dfrac{\sin^n x}{\sin^n x + \cos^n x}\, dx \qquad \qquad \text{...(2)}$$

adding (1) and (2), we get $2I = \int_{\pi/6}^{\pi/3} 1.\, dx = x \Big|_{\pi/6}^{\pi/3} = \dfrac{\pi}{3} - \dfrac{\pi}{6}$

$$= \dfrac{\pi}{6}$$

Hence $I = \dfrac{\pi}{12}$

Property :

$$\int_{-a}^{a} f(x)\, dx = \begin{cases} 0, & \text{if } f(x) \text{ is odd i.e., } f(-x) = -f(x) \\ 2\int_{0}^{a} f(x)\, dx, & \text{if } f(x) \text{ is even i.e., } f(-x) = f(x) \end{cases}$$

Example : Evaluate :

$$\int_{-\sqrt{2}}^{\sqrt{2}} \dfrac{(2x^7 + 3x^6 - 10x^5 - 7x^3 - 12x^2 + x + 1)}{(x^2 + 2)}\, dx$$

Solution. Let $I = \int_{-\sqrt{2}}^{\sqrt{2}} \dfrac{(2x^7 + 3x^6 - 10x^5 - 7x^3 - 12x^2 + x + 1)}{(x^2 + 2)}\, dx$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \dfrac{(2x^7 - 10x^5 - 7x^3 + x)}{(x^2 + 2)}\, dx + \int_{-\sqrt{2}}^{\sqrt{2}} \dfrac{(3x^6 - 12x^2 + 1)}{(x^2 + 2)}\, dx$$

$$= 0 + 2\int_{0}^{\sqrt{2}} \dfrac{3x^6 - 12x^2 + 1}{(x^2 + 2)}\, dx \qquad \{\because \text{ 1st member is odd and second member is even}\}$$

$$= 2\int_{0}^{\sqrt{2}} \dfrac{3x^2(x^2 + 2)(x^2 - 2) + 1}{(x^2 + 2)}\, dx$$

$$= 2\int_{0}^{\sqrt{2}} \left(3x^4 - 6x^2 + \dfrac{1}{x^2 + 2}\right) dx$$

$$= 6\dfrac{x^5}{5} - 12\dfrac{x^3}{3} + \dfrac{2}{\sqrt{2}}\tan^{-1}\left(\dfrac{x}{\sqrt{2}}\right)\Big|_{0}^{\sqrt{2}}$$

$$= \dfrac{24}{5}\sqrt{2} - 8\sqrt{2} + \sqrt{2} \cdot \dfrac{\pi}{4}$$

$$= \dfrac{\pi\sqrt{2}}{4} - \dfrac{16\sqrt{2}}{5}$$

$$= \frac{\sqrt{2}}{20}(5\pi - 64) = \frac{1}{10\sqrt{2}}(5\pi - 64)$$

Property :

$$\int_0^{2a} f(x)dx = \begin{cases} 0 \text{ if } f(2a - x) = -f(x) \\ 2\int_0^a f(x)dx \text{ if } f(2a - x) = f(x) \end{cases}$$

Example : Evaluate : $\int_0^{\pi/2} \ln \sin x \, dx$

Solution. Let $I = \int_0^{\pi/2} \ln \sin x \, dx$ ...(1)

$$= \int_0^{\pi/2} \ln \sin\left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/2} \ln \cos x \, dx \qquad\qquad ...(2)$$

Adding (1) and (2), $2I = \int_0^{\pi/2} \ln (\sin x \cos x) \, dx$

$$= \int_0^{\pi/2} \ln\left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\pi/2} \ln \sin 2x \, dx - \int_0^{\pi/2} \ln 2 \, dx$$

$$= \int_0^{\pi/2} \ln \sin 2x \, dx - \frac{\pi}{2} \ln 2$$

Put $2x = t \Rightarrow dx = \dfrac{dt}{2}$

When $x = 0 \Rightarrow t = 0$

$\qquad x = \pi/2 \Rightarrow t = \pi$

$\therefore \qquad 2I = \dfrac{1}{2}\int_0^{\pi} \ln \sin t \, dt - \dfrac{\pi}{2} \ln 2$

Here if $f(t) = \ln \sin t$

$\qquad f(\pi - t) = \ln \sin (\pi - t) = \ln \sin t = f(t)$

then $\qquad 2I = \dfrac{1}{2} \cdot 2\int_0^{\pi/2} \ln \sin t \, dt - \dfrac{\pi}{2} \ln 2$

$\qquad 2I = \int_0^{\pi/2} \ln \sin x \, dx - \dfrac{\pi}{2} \ln 2$

$\qquad\qquad = I - \dfrac{\pi}{2} \ln 2$

$\therefore \qquad I = -\dfrac{\pi}{2} \ln 2$

Hence remember that :

(i) $\displaystyle\int_0^{\pi/2} \ln \sin x \, dx = \int_0^{\pi/2} \ln \cos x \, dx = -\frac{\pi}{2}\ln 2$

(ii) $\displaystyle\int_0^{\pi/2} \ln \tan x \, dx = \int_0^{\pi/2} \ln \cot x \, dx = 0$

(iii) $\displaystyle\int_0^{\pi/2} \ln \sec x \, dx = \int_0^{\pi/2} \ln \mathrm{cosec}\, x \, dx = \frac{\pi}{2}\ln 2$

Example : Show that $\displaystyle\int_0^{\pi/2} f(\sin 2x)\sin x \, dx = \int_0^{\pi/2} f(\sin 2x)\cos x \, dx$

$$= \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x \, dx$$

Solution. Let I $= \displaystyle\int_0^{\pi/2} f(\sin 2x)\sin x \, dx$ ...(1)

$$= \int_0^{\pi/2} f\left(\sin 2\left(\frac{\pi}{2}-x\right)\right)\sin\left(\frac{\pi}{2}-x\right)dx$$

$$= \int_0^{\pi/2} f(\sin(\pi-2x))\cos x \, dx$$

$$= \int_0^{\pi/2} f(\sin 2x)\cos x \, dx$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} f(\sin 2x)(\sin x + \cos x)\, dx$$

$$= \sqrt{2}\int_0^{\pi/2} f(\sin 2x)\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)dx$$

$$= \sqrt{2}\int_0^{\pi/2} f(\sin 2x)\sin(x+\pi/4)\, dx$$

Put $x = \dfrac{\pi}{4}-t \Rightarrow dx = -\,dt$

When x = 0 $\Rightarrow t = \pi/4$

  x = π/2 $\Rightarrow t = -\pi/4$

∴ $2I = \sqrt{2}\displaystyle\int_{\pi/4}^{-\pi/4} f\left(\sin\left(\frac{\pi}{2}-2t\right)\right)\sin\left(\frac{\pi}{2}-t\right)(-dt)$

$$= \sqrt{2}\int_{-\pi/4}^{\pi/4} f(\cos 2t)\cos t \, dt$$

$$= \sqrt{2}\int_{-\pi/4}^{\pi/4} f(\cos 2x)\cos x \, dx$$

$$= 2\sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x \, dx$$

Hence I $= \sqrt{2}\displaystyle\int_0^{\pi/4} f(\cos 2x)\cos x \, dx$

Property : (Time Saving Property or Help line property)

$$\int_a^b f(x)\,dx = \int_{a-c}^{b-c} f(x+c)\,dx$$

and

$$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx$$

Analytical Method :

Put   x + c = t   ∴   dx = dt

When x = a – c ⇒ t = a

x = b – c ⇒ t = b

then $\int_{a-c}^{b-c} f(x+c)\,dx = \int_a^b f(t)\,dt = \int_a^b f(x)\,dx$

Also put x – c = t

∴   dx = dt

When x = a + c ⇒ t = a

x = b + c ⇒ t = b

then $\int_{a+c}^{b+c} f(x-c)\,dx = \int_a^b f(t)\,dt = \int_a^b f(x)\,dx$

Example : Show that the sum of two integrals

$$\int_{-4}^{-5} e^{(x+5)^2}\,dx + 3\int_{1/3}^{2/3} e^{9(x-2/3)^2}\,dx$$

is zero.

Solution. Let I = $\int_{-4}^{-5} e^{(x+5)^2}\,dx + 3\int_{1/3}^{2/3} e^{9(x-2/3)^2}\,dx$

$= \int_{-4+5}^{-5+5} e^{(x-5-5)^2}\,dx + 3\int_{\frac{1}{3}-\frac{2}{3}}^{\frac{2}{3}-\frac{2}{3}} e^{9\,(x+(2/3)-(2/3))^2}\,dx$

$= \int_1^0 e^{x^2}\,dx + 3\int_{-1/3}^0 e^{9x^2}\,dx$

Put 3x = – t in second integral then

3 dx = – dt

When x = – 1/3 ⇒ t = 1

x = 0 ⇒ t = 0

then $I = -\int_0^1 e^{x^2}\,dx - \int_1^0 e^{t^2}\,dt$

$= -\int_0^1 e^{x^2}\,dx + \int_0^1 e^{t^2}\,dt$

$= -\int_0^1 e^{x^2}\,dx + \int_0^1 e^{x^2}\,dx = 0$

Example : If $f(x) = \int_0^x \dfrac{e^t}{t}\,dt$, $x > 0$ then prove that $\int_1^x \dfrac{e^t\,dt}{(t+a)} = e^{-a}\{f(x+a) - f(1+a)\}$.

Solution. R.H.S. = $e^{-a}\{f(x+a) - f(1+a)\}$

$$= e^{-a} \left\{ \int_0^{x+a} \frac{e^t}{t} dt - \int_0^{1+a} \frac{e^t}{t} dt \right\}$$

$$= e^{-a} \left\{ \int_0^{x+a} \frac{e^t}{t} dt + \int_{1+a}^0 \frac{e^t}{t} dt \right\}$$

$$= e^{-a} \left\{ \int_{1+a}^0 \frac{e^t}{t} dt + \int_0^{x+a} \frac{e^t}{t} dt \right\}$$

$$= e^{-a} \left\{ \int_{1+a}^{x+a} \frac{e^t}{t} dt \right\}$$

$$= \int_{1+a}^{x+a} \frac{e^{t-a}}{t} dt$$

put $t - a = u \Rightarrow dt = du$

at $t = 1 + a$ we get $u = 1$

at $t = x + a$ we get $u = x$

$$= \int_1^x \frac{e^u}{(u+a)} du$$

$$= \int_1^x \frac{e^t}{(t+a)} dt$$

= L.H.S.

Property : (Time Saving property or Help line property)

$$\int_a^b f(x) \, dx = k \int_{a/k}^{b/k} f(kx) \, dx$$

and $$\int_a^b f(x) \, dx = \frac{1}{k} \int_{ak}^{bk} f\left(\frac{x}{k}\right) dx$$

Example : Evaluate : $\int_0^1 |\sin 2\pi x| \, dx$.

Solution. Let $2\pi x = t \Rightarrow 2\pi \, dx = dt \Rightarrow dx = \dfrac{dt}{2}$

if $x = 0 \Rightarrow t = 0$; $x = 1 \Rightarrow t = 2\pi$

$$\Rightarrow \quad I = \int_0^{2\pi} |\sin t| \frac{dt}{2\pi} = \frac{1}{2\pi} \int_0^{2\pi} |\sin t| \, dt$$

$$= \frac{1}{2\pi} \int_0^{\pi} \sin t \, dt - \frac{1}{2\pi} \int_{\pi}^{2\pi} \sin t \, dt$$

$$= -\frac{1}{2\pi} (\cos t)_0^{\pi} + \frac{1}{2\pi} (\cos t)_{\pi}^{2\pi}$$

$$= -\frac{1}{2\pi}(\cos\pi - \cos 0) + \frac{1}{2\pi}(\cos 2\pi - \cos\pi)$$

$$= -\frac{1}{2\pi}(-1-1) + \frac{1}{2\pi}(1-(-1)) = \frac{2}{\pi}$$

Example : Show that $\int_0^x e^{zx} e^{-z^2} dz = e^{x^2/4} \int_0^x e^{-z^2/4} dz.$

Solution. Let $I = \int_0^x e^{zx} e^{-z^2} dz$ (here x is constant)

put $z = \dfrac{x+t}{2}$ $\quad\therefore\quad$ $dz = \dfrac{1}{2} dt$

when $z = 0 \Rightarrow t = -x$

$z = x \Rightarrow t = x$

$\therefore \quad I = \dfrac{1}{2}\int_{-x}^x e^{\{x(x+t)/2\}} e^{\{-(x+t)^2/4\}} dt$

$$= \frac{1}{2}\int_{-x}^x e^{(x^2-t^2)/4} dt$$

$$= \frac{1}{2} e^{x^2/4}\int_{-x}^x e^{-t^2/4} dt$$

$$= \frac{1}{2} e^{x^2/4} \, 2\int_0^x e^{-t^2/4} dt \qquad\qquad (\because \text{ Function is even})$$

$$= e^{x^2/4}\int_0^x e^{-z^2/4} dt$$

OR

L.H.S. $= \int_0^x e^{zx} e^{-z^2} dz$

$$= \int_0^x e^{-(z^2-zx)} dz$$

$$= \int_0^x e^{-\{(z^2-zx+(x^2/4)-(x^2/4)\}} dz$$

$$= \int_0^x e^{-\{(z-x/2)^2-(x^2/4)\}} dz$$

$$= e^{x^2/4}\int_0^x e^{-\{(z-(x/2))^2\}} dz$$

put $z - x/2 = t \Rightarrow dz = dt$

at $z = 0$ we get $t = -x/2$

at $z = x$ we get $t = x/2$

$$= e^{x^2/4}\int_{-x/2}^{x/2} e^{-t^2} dt$$

put $t = u/2 \Rightarrow dt = du/2$

at   t = -x/2 we get u = -x

at   t = x/2 we get u = x

$$= \frac{1}{2} e^{x^2/4} \int_{-x}^{x} e^{-(u/2)^2} dz$$

$$= \frac{1}{2} 2 e^{x^2/4} \int_{0}^{x} e^{-u^2/4} dz \qquad (\because \text{ function is even})$$

$$= e^{x^2/4} \int_{0}^{x} e^{-z^2/4} dz$$

= R.H.S.

Property : (Reflection Property)

$$\int_{a}^{b} f(x) dx = \int_{-b}^{-a} f(-x) \, dx$$

Proof. Analytical Method :

Let $I = \int_{-b}^{-a} f(-x) \, dx$ ...(1)

Put $-x = t$ $\therefore$ $dx = -dt$

when $x = -b \Rightarrow t = b$

            $x = -a \Rightarrow t = a$

then $I = -\int_{b}^{a} f(t) \, dt$

$$= \int_{a}^{b} f(t) \, dt$$

$$= \int_{a}^{b} f(x) \, dx \qquad\qquad\qquad\qquad\qquad\qquad ...(2)$$

From (1) and (2), we get $\int_{a}^{b} f(x) \, dx = \int_{-b}^{-a} f(-x) \, dx$

Graphical Method

The curve y = f(–x) is obtained with the help of y = f(x), $\because$ y = f(–x) is the mirror image of y = f(x) in the y-axis.

It is clear from the Fig..
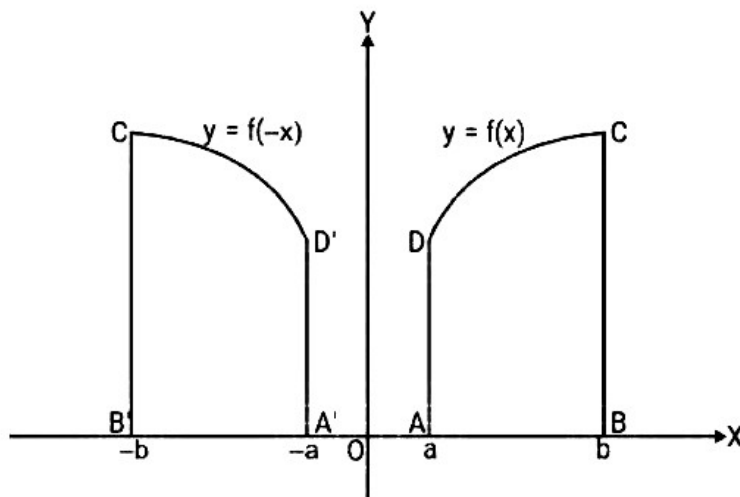
Fig.

Area of ABCDA = Area of BA'C'D'A

then $\quad \int_a^b f(x)\, dx = \int_{-b}^{-a} f(-x)\, dx$

Example : Evaluate : $\int_{-\pi/3}^{-\pi/6} \dfrac{dx}{1 + \tan^{2n} x}$.

Solution. Let $I = \int_{-\pi/3}^{-\pi/6} \dfrac{dx}{1 + \tan^{2n} x}$

$\qquad = \int_{\pi/6}^{\pi/3} \dfrac{dx}{1 + (\tan(-x))^{2n}}$

$\qquad = \int_{\pi/6}^{\pi/3} \dfrac{\cos^{2n} x\, dx}{(\sin^{2n} x + \cos^{2n} x)}$ ...(1)

and $\quad I = \int_{\pi/6}^{\pi/3} \dfrac{\cos^{2n}\left(\dfrac{\pi}{3} + \dfrac{\pi}{6} - x\right) dx}{\sin^{2n}\left(\dfrac{\pi}{3} + \dfrac{\pi}{6} - x\right) + \cos^{2n}\left(\dfrac{\pi}{3} + \dfrac{\pi}{6} - x\right)}$

$\qquad = \int_{\pi/6}^{\pi/3} \dfrac{\sin^{2n} x\, dx}{\cos^{2n} x + \sin^{2n} x}$ ...(2)

Adding (1) & (2) we get $\quad 2I = \int_{\pi/6}^{\pi/3} 1 \cdot dx = \dfrac{\pi}{3} - \dfrac{\pi}{6} = \dfrac{\pi}{6}$

$\therefore \quad I = \dfrac{\pi}{12}$

Property : If $f(x)$ is discontinuous at $x = a$ then

$\qquad \int_0^{2a} f(x)\, dx = \int_0^a \{f(a - x) + f(a + x)\}\, dx$

Proof : R.H.S. $= \int_0^a \{f(a - x) + f(a + x)\} dx$

$\qquad = \int_0^a f(a - x)\, dx + \int_0^a f(a + x)\, dx$

$\qquad = \int_0^a f(a - (a - x))\, dx + \int_{0+a}^{a+a} f(x)\, dx$

$\qquad = \int_0^a f(x)\, dx + \int_a^{2a} f(x)\, dx$

$\qquad = \int_0^{2a} f(x)\, dx = \text{L.H.S.}$

Example : Evaluate : $\int_0^{\pi} \dfrac{x \sin 2x \sin\left(\dfrac{\pi}{2} \cos x\right)}{(2x - \pi)}\, dx$

Solution. Let I = $\int_0^\pi \dfrac{x \sin 2x \sin\left(\dfrac{\pi}{2}\cos x\right)}{(2x-\pi)}dx$

∴ Integrand is discontinuous at x = π/2 then

$I = \int_0^{\pi/2} \dfrac{\left(\dfrac{\pi}{2}-x\right)\sin 2\left(\dfrac{\pi}{2}-x\right)\sin\left(\dfrac{\pi}{2}\cos\left(\dfrac{\pi}{2}-x\right)\right)}{2\left(\dfrac{\pi}{2}-x\right)-\pi}dx$

$+\int_0^{\pi/2} \dfrac{\left(\dfrac{\pi}{2}+x\right)\sin 2\left(\dfrac{\pi}{2}+x\right)\sin\left(\dfrac{\pi}{2}\cos\left(\dfrac{\pi}{2}+x\right)\right)}{2\left(\dfrac{\pi}{2}+x\right)-\pi}$

$=\int_0^{\pi/2} \dfrac{\left(\dfrac{\pi}{2}-x\right)\sin 2x.\sin\left(\dfrac{\pi}{2}\sin x\right)}{-2x}dx + \int_0^{\pi/2} \dfrac{\left(\dfrac{\pi}{2}+x\right)(-\sin 2x)\sin\left(\dfrac{\pi}{2}(-\sin x)\right)}{2x}$

$=\int_0^{\pi/2} \dfrac{\left(x-\dfrac{\pi}{2}\right)\sin 2x.\sin\left(\dfrac{\pi}{2}\sin x\right)}{2x}dx + \int_0^{\pi/2} \dfrac{\left(\dfrac{\pi}{2}+x\right)(\sin 2x)\sin\left(\dfrac{\pi}{2}(\sin x)\right)}{2x}$

$=\int_0^{\pi/2} \dfrac{2x \sin 2x \sin\left(\dfrac{\pi}{2}\sin x\right)}{2x}dx$

$=\int_0^{\pi/2} 2\sin x \cos x \sin\left(\dfrac{\pi}{2}\sin x\right)dx$

Put $\dfrac{\pi}{2}\sin x = t$ then $\cos x\, dx = \dfrac{2}{\pi}dt$

∴ $I = \dfrac{2}{\pi}\int_0^{\pi/2} 2\left(\dfrac{2t}{\pi}\right)\sin t\, dt$

$=\dfrac{8}{\pi^2}\int_0^{\pi/2} t \sin t\, dt$

$=\dfrac{8}{\pi^2}\left\{t(-\cos t)\Big|_0^{\pi/2} + \sin t\Big|_0^{\pi/2}\right\}$

$=\dfrac{8}{\pi^2}\{1-0\} = \dfrac{8}{\pi^2}$

Property :

$\int_a^b f(x)\, dx = (b-a)\int_0^1 f((b-a)x+a)dx$

Example : Given $\int_0^1 \dfrac{\sin t}{1+t} dt = \alpha$, find the value of $\int_{4\pi-2}^{4\pi} \dfrac{\sin(t/2)}{4\pi+2-t} dt$ in terms of α.

Solution. Let I $= \int_{4\pi-2}^{4\pi} \dfrac{\sin(t/2)}{4\pi+2-t} dt$

Where $a = 4\pi - 2$, $b = 4\pi$, $f(t) = \dfrac{\sin t/2}{4\pi+2-t}$

according to property - $\int_a^b f(x)\ dx = (b-a)\int_0^1 f((b-a)x + a)dx$

$$I = (4\pi - (4\pi - 2))\int_0^1 \dfrac{\sin\left(\dfrac{(4\pi - (4\pi - 2))t + 4\pi - 2}{2}\right)}{4\pi + 2 - (4\pi - (4\pi - 2))t + 4\pi - 2)} dt$$

$$I = 2\int_0^1 \dfrac{\sin(2\pi + (t-1))}{(4-2t)} dt \qquad\qquad [\because \sin(2\pi + \theta) = \sin\theta]$$

$$= 2\int_0^1 \dfrac{\sin(t-1)dt}{(4-2t)}$$

$$= \int_0^1 \dfrac{\sin(t-1)}{(2-t)} dt \qquad\qquad \text{(substitute t = 1- t)}$$

$$= \int_0^1 \dfrac{\sin(1-t-1)}{2-(1-t)} dt$$

$$= \int_0^1 \dfrac{\sin(-t)}{(1+t)} dt \qquad\qquad [\because \sin(-\theta) = -\sin(\theta)]$$

$$= -\int_0^1 \dfrac{\sin t}{1+t} dt$$

$$= -\alpha \qquad\qquad \text{(given)}$$

Property :

$$\int_0^{nT} f(x)\ dx = n\int_0^T f(x)\ dx;$$

where 'T' is the period of the function and $n \in I$
i.e., $f(x + T) = f(x)$

Example : Evaluate : $\int_0^{100\pi} \sqrt{(1 + \cos 2x)}\ dx$

Solution. Let I $= \int_0^{100\pi} \sqrt{(1 + \cos 2x)}\ dx$

$$= \int_0^{100\pi} \sqrt{2}\ |\cos x|\ dx$$

$$= \sqrt{2}\int_0^{100\pi} |\cos x|\ dx$$

$$= 100\sqrt{2}\int_0^{\pi} |\cos x|\ dx \qquad\qquad (\because\ |\cos x|\ \text{is periodic with period } \pi)$$

$$= 100\sqrt{2}\left\{\int_0^{\pi/2} |\cos x|\, dx + \int_{\pi/2}^{\pi} |\cos x|\, dx\right\}$$

$$= 100\sqrt{2}\left\{\int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{\pi} \cos x\, dx\right\}$$

$$(\because \quad \cos x \text{ is +ve in } (0,\, \pi/2) \text{ and } -\text{ve is } (\pi/2,\, \pi))$$

$$= 100\sqrt{2}\left\{\sin x\Big|_0^{\pi/2} - \sin x\Big|_{\pi/2}^{\pi}\right\}$$

$$= 100\sqrt{2}\{(1-0)-(0-1)\}$$

$$= 200\sqrt{2}$$

Property : $\int_{mT}^{nT} f(x)\, dx = (n-m)\int_0^T f(x)\, dx$

where 'T' is the period of the function and m, n $\in$ I

i.e., $\quad f(x + T) = f(x)$

Proof. L.H.S. $= \int_{mT}^{nT} f(x)\, dx = \int_0^{(n-m)T} f(x + mT)\, dx$

$$= \int_0^{(n-m)T} f(x)\, dx \qquad\qquad (\because \quad f(x) \text{ is periodic})$$

$$= (n-m)\int_0^T f(x)\, dx$$

$$= \text{R.H.S.}$$

Example : Evaluate : $\int_0^{199\pi} \sqrt{\left(\dfrac{1-\cos 2x}{2}\right)}\, dx$

Solution. Let I $= \int_{-\pi}^{199\pi} \sqrt{\left(\dfrac{1-\cos 2x}{2}\right)}\, dx$

$$= \int_{-\pi}^{199\pi} |\sin x|\, dx$$

$$= (199-(-1))\int_0^{\pi} |\sin x|\, dx \qquad (\because \quad |\sin x| \text{ is periodic with period } \pi)$$

$$= 200\int_0^{\pi} \sin x\, dx$$

$$= 200\, (-\cos x)\Big|_0^{\pi}$$

$$= 200\,(1-(-1)) = 400$$

Property :

$$\int_{a+nT}^{b+nT} f(x)\, dx = \int_a^b f(x)\, dx$$

where 'T' is the period of the function and n $\in$ I

i.e., $\quad f(x + T) = f(x)$

Proof. L.H.S. $= \int_{a+nT}^{b+nT} f(x)\, dx = \int_a^b f(x + nT)\, dx$

$$= \int_a^b f(x)\, dx \qquad\qquad (\because \quad f(x + nT) = f(x))$$

$$= \text{R.H.S.}$$

Example : Show that

$$\int_0^{n\pi + V} |\sin x|\, dx = (2n+1) - \cos V$$

where n $\in$ N and $0 \le V \le \pi$.

Solution. Let $I = \int_0^{n\pi + V} |\sin x| \, dx$

$= \int_0^{n\pi} |\sin x| \, dx + \int_{n\pi}^{n\pi + V} |\sin x| \, dx$

$= n \int_0^{\pi} |\sin x| \, dx + \int_0^{V} |\sin x| \, dx$ $\qquad (\because \ |\sin x| \text{ is periodic with period } \pi)$

$= n \int_0^{\pi} \sin x \, dx + \int_0^{V} \sin x \, dx$ $\qquad (\because \ 0 \le V \le \pi)$

$= n(-\cos x)\big|_0^{\pi} + (-\cos x)\big|_0^{V}$

$= n(1 - (-1)) + (-\cos V + 1)$

$= (2n + 1) - \cos V$

Ex. Find the value of $I = \int_0^{5\pi + \frac{\pi}{3}} |\sin x| \, dx$

Sol. $I = \int_0^{5\pi} |\sin x| \, dx + \int_{5\pi}^{5\pi + \frac{\pi}{3}} |\sin x| \, dx$

$= 5 \int_0^{\pi} |\sin x| \, dx + \int_0^{\pi/3} \sin x \, dx$

$= 5(-\cos x)\big|_0^{\pi} + (-\cos x)\big|_0^{\pi/3}$

$= 5(1+1) - \left(\dfrac{1}{2} - 1 \right) = \dfrac{21}{2}$

Property : Leibnitz's Rule :

(i)   If $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in [a, b] and $f(x, t)$ is continuous then

$$\frac{d}{dx} \left\{ \int_{u(x)}^{v(x)} f(x, t) \, dt \right\} = \int_{u(x)}^{v(x)} \frac{\partial}{\partial x} f(x, t) \, dt + \frac{dv(x)}{dx} f(x, v(x)) - \frac{du(x)}{dx} f(x, u(x))$$

Note : Where $\dfrac{\partial}{\partial x} f(x, t)$ represents the differentiation with respect to x treating t as constant.

(ii)   If f is continuous function on [a, b] and $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in [a, b] then

$$\frac{d}{dx} \left\{ \int_{u(x)}^{v(x)} f(x) \, dt \right\} = f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}$$

Example : If $\int_{\pi/3}^{x} \sqrt{(3 - \sin^2 t)} \, dt + \int_0^{y} \cos t \, dt = 0$ then evaluate $\dfrac{dy}{dx}$.

Solution. We have $\int_{\pi/3}^{x} \sqrt{(3 - \sin^2 t)} \, dt + \int_0^{y} \cos t \, dt = 0$

Differentiating both sides w.r.t. x then

$$\frac{d}{dx} \int_{\pi/3}^{x} \sqrt{(3 - \sin^2 t)} \, dt + \frac{d}{dx} \int_0^{y} \cos t \, dt = 0$$

$$\Rightarrow \quad \sqrt{(3 - \sin^2 x)} \frac{d(x)}{dx} - \sqrt{(3 - \sin^2 \pi/3)} \frac{d(\pi/3)}{dx} + \cos y \frac{dy}{dx} - \cos 0 \frac{d}{dx}(0) = 0$$

$$\Rightarrow \quad \sqrt{(3 - \sin^2 x)} . 1 - 0 + \cos y \frac{dy}{dx} - 0 = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\sqrt{(3-\sin^2 x)}}{\cos y}$$

Property :

$$\int_a^{a+T} f(x)\, dx \text{ is independent of } a$$

where 'T' is the period of the function

i.e., $\quad f(x+T) = f(x)$

Proof. Let $I = \int_a^{a+T} f(x)\, dx$

$$\therefore \quad \frac{dI}{da} = f(a+T)\frac{d}{da}(a+T) - f(a)\frac{da}{da}$$

$$= f(a+T)\cdot 1 - f(a)\cdot 1$$

$$= f(a+T) - f(a)$$

$$= 0 \qquad\qquad \{\because\ f(a+T) = f(a)\}$$

Thus, $I = c$ (independent of a)

Hence $\int_a^{a+T} f(x)\, dx$ is independent of a.

Example : Evaluate : $\int_a^{a+\pi/2}(\sin^4 x + \cos^4 x)dx$.

Solution. $\because\ \sin^4 x + \cos^4 x$ is periodic with period $\pi/2$ then $\int_a^{a+\pi/2}(\sin^4 x + \cos^4 x)\, dx$

$$= \int_0^{\pi/2}(\sin^4 x + \cos^4 x)\, dx$$

$$= \int_0^{\pi/2}\{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x\}\, dx$$

$$= \int_0^{\pi/2}\left(1 - \frac{1}{2}\sin^2 2x\right)dx$$

$$= \int_0^{\pi/2}\left(1 - \frac{1}{4}(1 - \cos 4x)\right)dx$$

$$= \frac{1}{4}\int_0^{\pi/2}(3 + \cos 4x)\, dx$$

$$= \frac{1}{4}\left(3x + \frac{\sin 4x}{4}\right)\Big|_0^{\pi/2}$$

$$= \frac{1}{4}\left\{\left(\frac{3\pi}{2} + 0\right) - 0\right\}$$

$$= \frac{3\pi}{8}$$

Property :

Suppose that $f(x, \alpha)$ and $f_\alpha'(x, \alpha)$ are continuous functions when $c \le \alpha \le d$ and $a \le x \le b$ then

$$I'(\alpha) = \int_a^b f_\alpha'(x, \alpha)dx \text{ where } I'(\alpha) \text{ is}$$

the derivative of $I(\alpha)$ w.r.t. α and $f_\alpha{'}(x, \alpha)$ is the derivative of $f(x, \alpha)$ treating x as constant.

Example : Evaluate : $\int_0^1 \dfrac{x^a - 1}{\ln x} dx$

Solution. Let $I(a) = \int_0^1 \dfrac{x^a - 1}{\ln x} dx$

Where $f_a = \dfrac{x^a - 1}{\ln x}$,

then $\dfrac{df_a}{da} = \dfrac{d}{da}\left(\dfrac{x^a - 1}{\ln x}\right)$

$\dfrac{df_a}{da} = \dfrac{1}{\ln x} \dfrac{d(x^a - 1)}{da}$

$\dfrac{df_a}{da} = \dfrac{1}{\ln x}\left[\dfrac{dx^a}{da} - \dfrac{d(1)}{da}\right]$

$\dfrac{df_a}{da} = \dfrac{1}{\ln x}(x^a \ln x - 0) = \dfrac{x^a \ln x}{\ln x} = x^a$

According to property - $I'(a) = \int_a^b f_a'(x, a)dx$ where $I'(a)$ is the derivative of $I(a)$

$\therefore \quad I'(a) = \dfrac{dI(a)}{da} = \int_0^1 x^a dx$

$= \dfrac{x^{a+1}}{a+1}\bigg|_0^1$

$\dfrac{dI(a)}{da} = \dfrac{1}{(a+1)}$

$dI(a) = \dfrac{da}{a+1}$

Integrating both sides w.r.t. a then

$\qquad I(a) = \ln(a + 1) + c$ ...(1)

for $\quad a = 0, \quad I(0) = \ln 1 + c$

$\qquad 0 = 0 + c \Rightarrow c = 0$

by equation (1)

$\therefore \quad I(a) = \ln(a + 1)$

Property :

If $f(t)$ is an odd function then $\phi(x) = \int_a^x f(t)\,dt$ is an even function.

Proof. Since $\phi(x) = \int_a^x f(t)\,dt$

$\therefore \quad \phi(-x) = \int_a^{-x} f(t)\,dt$

$$= \int_a^{-a} f(t)\,dt + \int_{-a}^{-x} f(t)\,dt$$

$$= \int_{-a}^{a} f(t)\,dt + \int_{-a}^{-x} f(t)\,dt$$

$$= 0 + \int_a^x f(-t)\,dt$$

$$= -\int_x^a f(-t)\,dt \qquad\qquad\qquad\qquad \{\because\ f(t)\ \text{is odd}\}$$

$$= \int_a^x f(t)\,dt$$

$$= \phi(x)$$

Hence $\phi(-x) = \phi(x)$.

Example : Prove that $F(x) = \int_0^x \ln\left(\dfrac{1-t}{1+t}\right) dt$ is an even function.

Solution. Let $f(t) = \ln\left(\dfrac{1-t}{1+t}\right)$

$$\therefore \qquad f(-t) = \ln\left(\frac{1+t}{1-t}\right)$$

$$= -\ln\left(\frac{1-t}{1+t}\right)$$

$$= -f(t)$$

$\Rightarrow \qquad f(-t) = -f(t) \qquad\qquad\qquad \therefore\ f(t)$ is an odd function.

Hence $F(x) = \int_0^x \ln\left(\dfrac{1-t}{1+t}\right) dt$ is an even function. [By using above property]

Property :

If $f(t)$ is an even function then $\phi(x) = \int_0^x f(t)\,dt$ is an odd function.

Proof. The proof is similar as above. Follow the same steps considering $f(t)$ to be an even function.

Example : Prove that $F(x) = \int_0^x \left(\dfrac{t}{e^t-1} + \dfrac{t}{2} + 1\right) dt$ is an odd function.

Solution. Let $f(t) = \dfrac{t}{e^t-1} + \dfrac{t}{2} + 1$

$$\therefore \qquad f(-t) = -\frac{t}{e^{-t}-1} - \frac{t}{2} + 1$$

$$= -\frac{te^t}{1-e^t} - \frac{t}{2} + 1$$

$$= \frac{te^t}{e^t-1} - \frac{t}{2} + 1$$

$$= t\left(1 + \frac{1}{e^t - 1}\right) - \frac{t}{2} + 1$$

$$= t + \frac{t}{e^t - 1} - \frac{t}{2} + 1$$

$$= \frac{t}{e^t - 1} + \frac{t}{2} + 1$$

$$= f(t)$$

$$\Rightarrow \quad f(-t) = f(t) \qquad\qquad \therefore \quad f(t) \text{ is an even function.}$$

Hence $F(x) = \int_0^x \left(\frac{t}{e^t - 1} + \frac{t}{2} + 1\right) dt$ is an odd function.

Important Note : If $f(t)$ an even function then $\phi(x) = \int_a^x f(t)\, dt$ is not necessarily an odd function for $a \neq 0$. It will be an odd function if $\int_0^a f(t)\, dt = 0$.

Proof. Suppose $\phi(x) = \int_a^x f(t)\, dt$ is an odd function then

$$\phi(-x) = -\phi(x)$$

$$\Rightarrow \quad \int_a^{-x} f(t)\, dt = -\int_a^x f(t)\, dt$$

$$\Rightarrow \quad \int_a^0 f(t)\, dt + \int_0^{-x} f(t)\, dt = -\left\{\int_a^0 f(t)\, dt + \int_0^x f(t)\, dt\right\}$$

$$\Rightarrow \quad -\int_0^a f(t)\, dt - \int_0^x f(t)\, dt = \int_0^a f(t)\, dt - \int_0^x f(t)\, dt$$

$$\Rightarrow \quad -\int_0^a f(t)\, dt - \int_0^x f(t)\, dt = \int_0^a f(t)\, dt - \int_0^x f(t)\, dt \qquad (\because \quad f(t) \text{ is an even})$$

$$\Rightarrow \quad 2\int_0^a f(t)\, dt = 0 \quad \text{or} \quad \int_0^a f(t) = 0$$

Property :

If on an interval [a, b] (a < b), and the functions f(x) and g(x) satisfy the condition $f(x) \geq g(x)$ then

$$\int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx$$

In Particular :

If $f(x) \geq 0$ then $\int_a^b f(x)\, dx \geq 0 \ \forall \ x \in [a, b]$

Proof : Graphical Method :

As we know that $\int_a^b f(x)dx$ denotes the area under f(x) from a to b and if $f(x) \geq g(x) \ \forall \ x \in$ (a, b) then clearly area under f(x) from a to b which can be seen by given fig.

It is clear from the Fig. Area of curvilinear trapezoid aBCb $\geq$ Area of curvilinear trapezoid aADb then $\int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx$ and if $\int_a^b g(x) = 0$ then $\int_a^b f(x)dx \geq 0$

Fig.
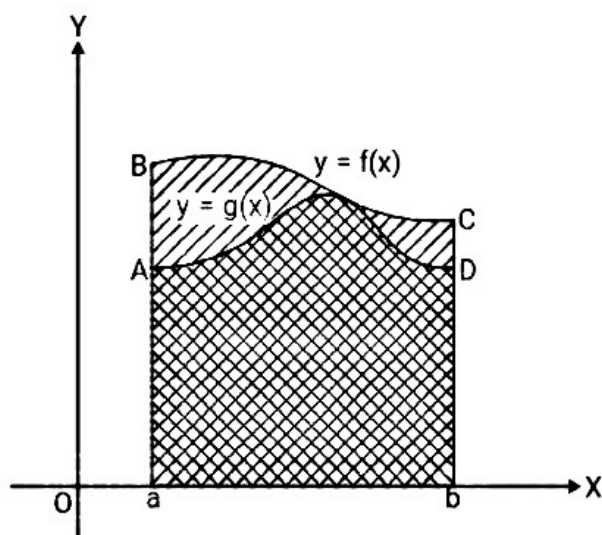
Example : Prove that :

$$\int_0^1 e^{-x} \cos^2 x \, dx < \int_0^1 e^{-x^2} \cos^2 x \, dx.$$

Solution. $\because \ 0 < x < 1$

then $\quad x > x^2 \qquad \Rightarrow \qquad -x < -x^2$

$$e^{-x} < e^{-x^2}$$

$$e^{-x} \cos^2 x < e^{-x^2} \cos^2 x$$

then $\quad \displaystyle\int_0^1 e^{-x} \cos^2 x \, dx < \int_0^1 e^{-x^2} \cos^2 x \, dx$

Property :

If at every point x of an interval [a, b] the inequalities
$$g(x) \le f(x) \le h(x)$$

are fulfilled then $\displaystyle\int_a^b g(x)dx \le \int_a^b f(x)dx \le \int_a^b h(x)dx, \ a < b$

Proof. Graphical Method :

It is clear from the Fig.

Area of curvilinear trapezoid aAFb

$\le$ Area of curvilinear trapezoid aBEb

$\le$ Area of curvilinear trapezoid aCDb

i.e., $\quad \displaystyle\int_a^b g(x)dx \le \int_a^b f(x)dx \le \int_a^b h(x)dx$

Fig.
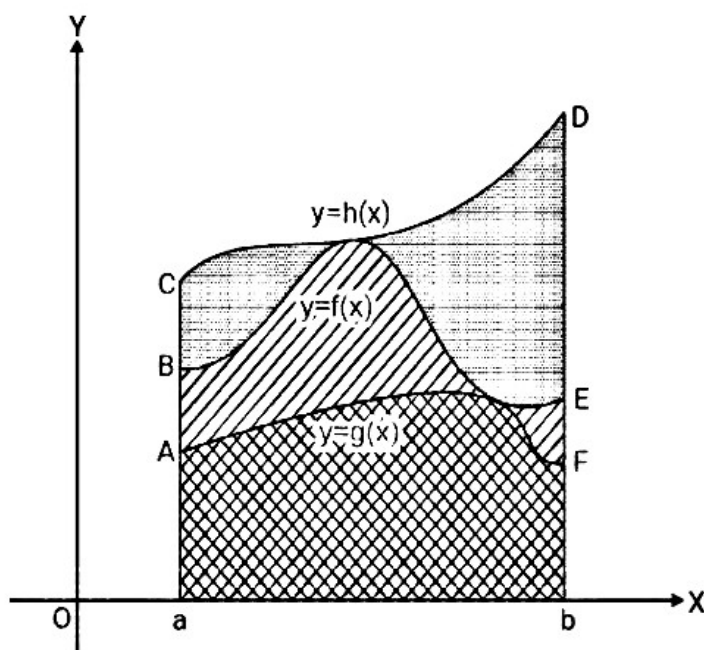
Example : Prove that :

$$0 < \int_0^1 \frac{x^7 dx}{\sqrt[3]{(1+x^8)}} < \frac{1}{8}$$

Solution. Since

$$0 < \frac{x^7}{\sqrt[3]{(1+x^8)}} < x^7 \ \forall \ 0 < x < 1$$

then
$$\int_0^1 0 \, dx < \int_0^1 \frac{x^7}{\sqrt[3]{(1+x^8)}} dx < \int_0^1 x^7 \, dx$$

Hence
$$0 < \int_0^1 \frac{x^7 \, dx}{\sqrt[3]{(1+x^8)}} < \frac{1}{8}$$

Property : If m is the least value (global minimum) and M is the greatest value (global maximum) of the function f(x) on the interval [a, b] (estimation of an integral). Then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

Proof : Analytical Method :

It is given that $m \le f(x) \le M$

then
$$\int_a^b m \, dx \le \int_a^b f(x) \, dx \le \int_a^b M \, dx$$

$$\Rightarrow \quad m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

Graphical Method :

It is clear from the Fig..
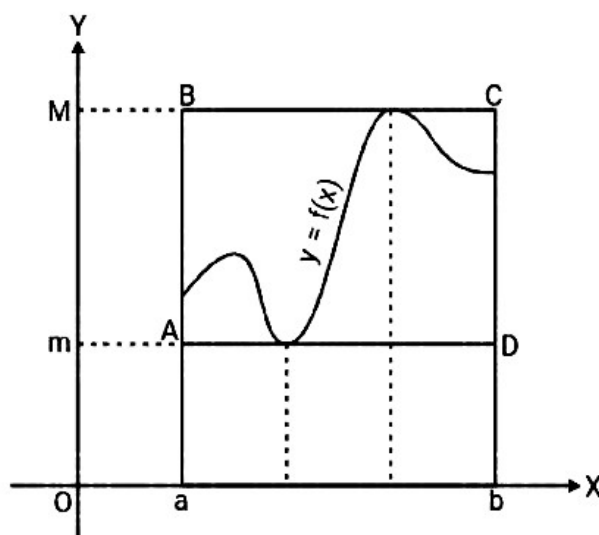
Fig.

Area of abDA $\leq \int_a^b f(x)\,dx \leq$ Area of aBCb

i.e., $m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a)$

Example : Prove that $1 < \int_0^2 \left(\dfrac{5-x}{9-x^2} \right) dx < \dfrac{6}{5}$

Solution. Let $f(x) = \dfrac{5-x}{9-x^2}$

$\therefore$ $f'(x) = -\dfrac{(x-9)(x-1)}{(9-x^2)^2}$

for critical points $f'(x) = 0$

$\therefore$ $x = 9$ and $x = 1$

$\because$ $x \in [0, 2])$ so we take $x = 1$.

then $f(0) = 5/9$

        $f(1) = 1/2$

        $f(2) = 3/5$

$\therefore$ The greatest and the least values of the integral in the interval [0, 2] are respectively, equal to $f(2) = 3/5$ and $f(1) = 1/2$. Hence

$$(2-0) \times \frac{1}{2} < \int_0^2 \left(\frac{5-x}{9-x^2} \right) dx < (2-0)\frac{3}{5}$$

or $1 < \int_0^2 \left(\dfrac{5-x}{9-x^2} \right) dx < \dfrac{6}{5}$

Property : The following inequality is valid :

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|\,dx \qquad (a \leq b)$$

or if a is not necessarily less than b, then

$$\left|\int_a^b f(x)dx\right| \le \left|\int_a^b |f(x)|\,dx\right|,$$

if f and | f | are integrable on [a, b].

Proof. Obviously, $- |f(x)| \le f(x) \le |f(x)|$, $\forall x \in [a, b]$

$\Rightarrow$ $\quad \int_a^b - |f(x)|\,dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|\,dx$ $\qquad\qquad$ (a < b)

or $\quad -\int_a^b |f(x)|\,dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|\,dx$

or $\quad \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

If b < a, then $\left|\int_a^b f(x)dx\right| = \left|\int_b^a f(x)dx\right| \le \int_b^a |f(x)|\,dx$

$$= \left|\int_a^b |f(x)|\,dx\right|$$

Hence $\left|\int_a^b f(x)dx\right| \le \left|\int_a^b |f(x)|\,dx\right|$

Example : Estimate the absolute value of the integral $\int_{10}^{19} \dfrac{\sin x}{1+x^8}dx$.

Solution. Since $|\sin x| \le 1$ for $x \ge 10$ then

$$\left|\frac{\sin x}{1+x^8}\right| \le \frac{1}{|1+x^8|} \qquad\qquad\qquad ...(1)$$

but $10 \le x \le 19$

$\therefore$ $\quad 1 + x^8 > x^8 \ge 10^8$

$\Rightarrow$ $\quad \dfrac{1}{1+x^8} < \dfrac{1}{x^8} \le \dfrac{1}{10^8}$

or $\quad \dfrac{1}{|1+x^8|} \le \dfrac{1}{10^8}$

from (1) & (2) we get $\qquad \left|\dfrac{\sin x}{1+x^8}\right| \le 10^{-8}$

$\therefore$ $\quad \left|\int_{10}^{19} \dfrac{\sin x}{1+8^8}dx\right| \le \int_{10}^{19} 10^{-8}dx$

$\qquad = (19 - 10) \times 10^{-8}$

$\qquad = 9 \times 10^{-8}$

$\qquad = (10 - 1) \times 10^{-8}$

$\qquad = 10^{-7} - 10^{-8} < 10^{-7}$

Hence $\left|\int_{10}^{19} \dfrac{\sin x}{1+x^8}dx\right| < 10^{-7}$

$\therefore$ $\quad$ The true value of the integral $\approx 10^{-8}$

Property :

If $f^2(x)$ and $g^2(x)$ are integrable on the interval [a, b], the **Schwarz-Bunyakovsky** inequality takes place :

$$\left| \int_a^b f(x) \, g(x) \, dx \right| \leq \sqrt{\left(\int_a^b f^2(x) \, dx \right) \left(\int_a^b g^2(x) \, dx \right)}$$

Proof. Let $F(x) = \{f(x) - \lambda g(x)\}^2 \geq 0$

$\therefore \qquad \int_a^b \{f(x) - \lambda g(x)\}^2 \, dx \geq 0$

$\Rightarrow \qquad \int_a^b \{\lambda^2 (g(x))^2 - 2\lambda f(x) \, g(x) + f^2(x)\} dx \geq 0$

$\Rightarrow \qquad \lambda^2 \int_a^b (g(x))^2 \, dx - 2\lambda \int_a^b f(x) \, g(x) dx + \int_a^b f^2(x) dx \geq 0$

$\therefore \qquad$ Discriminant is one positive i.e., $B^2 - 4AC \leq 0$

$\Rightarrow \qquad 4 \left\{ \int_a^b f(x) \, g(x) dx \right\}^2 \leq 4 \int_a^b f^2(x) dx \int_a^b g^2(x) \, dx$

Hence $\left| \int_a^b f(x) \, g(x) \, dx \right| \leq \sqrt{\int_a^b f^2(x) \, dx \, . \, \int_a^b g^2(x) dx}$

Example : Prove that $\int_0^1 \sqrt{(1+x)(1+x^3)} \, dx \leq \sqrt{\int_0^1 (1+x) dx \, . \, \int_0^1 (1+x^3) \, dx}$

$$= \sqrt{\left[x + \frac{x^2}{2} \right]_0^1 \left[x + \frac{x^4}{4} \right]_0^1}$$

$$= \sqrt{\left(\frac{3}{2} . \frac{5}{4} \right)}$$

$$= \sqrt{\left(\frac{15}{8} \right)}$$

Hence $\int_0^1 \sqrt{(1+x)(1+x^3)} \, dx \leq \sqrt{\left(\frac{15}{8} \right)}$

Property : If f(x) is continuous on [a, b] then there exists a point $c \in$ (a, b) such that

$$\int_a^b f(x) dx = f(c) \, (b - a)$$

Then number $f(c) = \dfrac{1}{(b-a)} \int_a^b f(x) \, dx$ is called the **Mean value** of the function f(x) on the interval [a, b].

Proof : Analytical Method :

For a < b. If m & M are smallest and greatest values of f(x) on [a, b]

then $\quad m(b - a) \leq \int_a^b f(x) dx \leq (b - a) M$

or $\qquad m \leq \dfrac{1}{(b-a)} \int_a^b f(x) dx \leq M$

Since f(x) is continuous on [a, b], It takes on all intermediate values between m and M. Therefore, some values f(c) (a ≤ f(c) ≤ b), we will have

or $\qquad \int_a^b f(x)dx = f(c)(b-a)$

Graphical Method :

It is clear from the Fig.
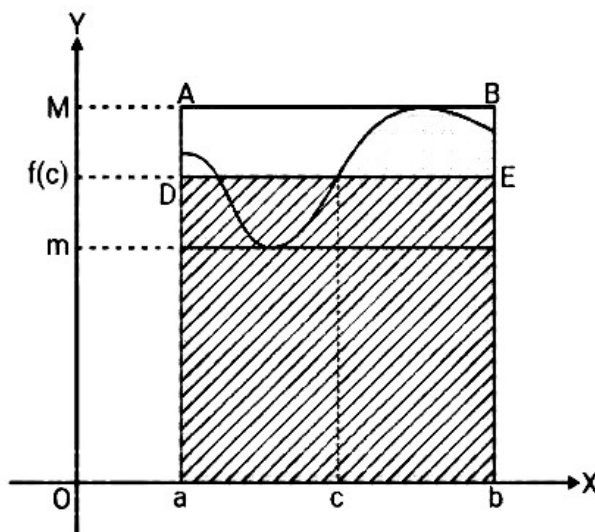
Fig.

Area of aABb = Area of aDEb

i.e., $\qquad \int_a^b f(x)dx = (b-a)f(c) \qquad$ or $\qquad f(c) = \dfrac{1}{(b-a)}\int_a^b f(x)\,dx$

Note : Used of this property also in physics if we evaluate average of physical quantities.

i.e., Average speed = $\dfrac{1}{v-u}\int_u^v v(t)dt$

Example : Find the mean value of x(a – x) over the range (0, a).

Solution. Mean value = $\dfrac{\int_0^a x(a-x)\,dx}{(a-0)}$

$= \dfrac{1}{a}\int_0^a (ax-x^2)dx$

$= \dfrac{1}{a}\left\{ \dfrac{ax^2}{2} - \dfrac{x^3}{3} \right\}_0^a$

$= \dfrac{1}{a}\left\{ \dfrac{a^3}{2} - \dfrac{a^3}{3} - 0 \right\} = \dfrac{a^2}{6}$

The **Root Mean Square Value (RMSV)** of a function y = f(x) the range (a, b) is

$$\sqrt{\left(\dfrac{\int_a^b y^2 dx}{(b-a)} \right)}$$

Example : Find RMSV of ln x over the range x = 1 to x = e

Solution. Here y = ln x

$$\text{RMSV} = \sqrt{\left(\frac{\int_1^e (\ln x)^2 \, dx}{(e-1)}\right)} \qquad \qquad ...(1)$$

Now $\int (\ln x)^2 . 1 \, dx = (\ln x)^2 . x - \int 2 \ln x \frac{1}{x} x \, dx$

$= x (\ln x)^2 - 2 \int \ln x \, dx$

$= x (\ln x)^2 - 2 \left\{ \ln x . x - \int \frac{1}{x} x \, dx \right\}$

$= x (\ln x)^2 - 2x \ln x + 2x$

$\therefore \quad \int_1^e (\ln x)^2 \, dx = x(\ln x)^2 - 2x \ln x + 2x \Big|_1^e$

$= (e - 2e + 2e) - (0 - 0 + 2)$

$= e - 2$

From (1), we get

$$\text{RMSV} = \sqrt{\left(\frac{e-2}{e-1}\right)}$$

Property :

$$\int_a^b f(x) \, d\alpha(x) + \int_a^b \alpha(x) \, df(x) = \alpha(b) f(b) - \alpha(a) f(a)$$

Example : Evaluate : $\int_0^5 (x^2 + 1) d\,[x]$, where [.] denotes the greatest integral function

Solution. Here f(x) = x² + 1 and α(x) = [x]

df(x) = 2xdx

According to property,

$$\int_a^b f(x) \, d\alpha(x) + \int_a^b \alpha(x) \, df(x) = \alpha(b) f(b) - \alpha(a) f(a)$$

$\therefore \quad \int_0^5 (x^2 + 1) \, d\,[x] + \int_0^5 [x] \, 2x \, dx = [5].(25 + 1) - 0$

$\Rightarrow \quad \int_0^5 (x^2 + 1) \, d[x] = 130 - 2\int_0^5 x[x] \, dx$

$= 130 - 2 \left\{ \int_0^1 0 \, dx + \int_1^2 x \, dx + \int_2^3 2x \, dx + \int_3^4 3x \, dx + \int_4^5 4x \, dx \right\}$

$= 130 - 2 \left\{ 0 + \frac{3}{2} + (9 - 4) + \frac{3}{2}(16 - 9) + 2(25 - 16) \right\}$

$= 130 - 70 = 60$

Property :

(i)   If the function f(x) increases and has a concave graph in the interval [a, b], then

$$(b-a)f(a) < \int_a^b f(x)dx < (b-a)\frac{f(a)+f(b)}{2}$$

(ii)   If the function f(x) increases and has a convex graph in the interval [a, b] then

$$(b-a)\frac{f(a)+f(b)}{2} < \int_a^b f(x)dx < (b-a)f(b)$$

Example : Prove that : $1 < \int_0^1 \sqrt{(1+x^4)}dx < \frac{1+\sqrt{2}}{2}$

Solution. Let $f(x) = \sqrt{(1+x^4)}$

$\therefore$ $f''(x) = \frac{2x^2(x^4+3)}{(1+x^4)^{3/2}} > 0, 0 < x < 1$

$\therefore$ graph is concave up.

$\therefore$ $(1-0)\,f(0) < \int_0^1 \sqrt{(1+x^4)}\,dx < (1-0)\frac{f(0)+f(1)}{2}$

$\Rightarrow$ $1 < \int_0^1 \sqrt{(1+x^4)}\,dx < \frac{1+\sqrt{2}}{2}$

Property   : Improper Integrals :

(i) $\int_a^\infty f(x)\,dx = \underset{b\to+\infty}{Lim}\int_a^b f(x)\,dx$

(ii) $\int_{-\infty}^a f(x)\,dx = \underset{a\to-\infty}{Lim}\int_\alpha^a f(x)\,dx$

(iii) $\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_c^{+\infty} f(x)\,dx = \underset{a\to-\infty}{\lim}\int_\alpha^a f(x)dx + \underset{b\to\infty}{\lim}\int_c^b f(x)dx$

Example : Prove that : $\int_0^\infty \frac{dx}{(x+\sqrt{(x^2+1)})^n} = \frac{n}{n^2-1}(n > 1)$

Solution. Put In $(x+\sqrt{(x^2+1)}) = t$

or $x + \sqrt{(x^2+1)} = e^t$ $\Rightarrow$ $\sqrt{(x^2+1)} = e^t - x$

or $x^2 + 1 = e^{2t} - 2xe^t + x^2$ $\therefore$ $x = \frac{e^{2t}-1}{2e^t} = \frac{e^t - e^{-t}}{2}$

$\therefore$ $dx = \left(\frac{e^t + e^{-t}}{2}\right)dt$

when $x \to 0, t \to 0$ and $x \to \infty, t \to \infty$

L.H.S. $= \int_0^\infty \frac{(e^t + e^{-t})}{2.e^{nt}}dt$

$= \frac{1}{2}\int_0^\infty (e^{(1-n)t} + e^{-(1+n)t})dt$

$$= \frac{1}{2} \underset{a \to +\infty}{\text{Lim}} \left\{ \frac{e^{(1-n)t}}{(1-n)} - \frac{e^{-(1+n)t}}{(1+n)} \right\}_0^a$$

$$= \frac{1}{2} \underset{a \to \infty}{\text{Lim}} \left\{ \frac{1}{(1-n)e^{(n-1)a}} - \frac{1}{(n+1)e^{(n-1)a}} - \frac{1}{1-n} + \frac{1}{1+n} \right\}$$

$$= \frac{1}{2} \left\{ 0 - 0 + \frac{1}{n-1} + \frac{1}{n+1} \right\}$$

$$= \frac{1}{2} \left\{ \frac{2n}{n^2-1} \right\} = \frac{n}{n^2-1}$$

Definite Integral as the limit of a sum (Integration by First Principle Rule)

Let $f(x)$ be a single valued continuous function in the interval (a, b), (a < b) and if the interval (a, b) be divided into n equal parts, each of width h, we have

$$b - a = nh$$

Fig.
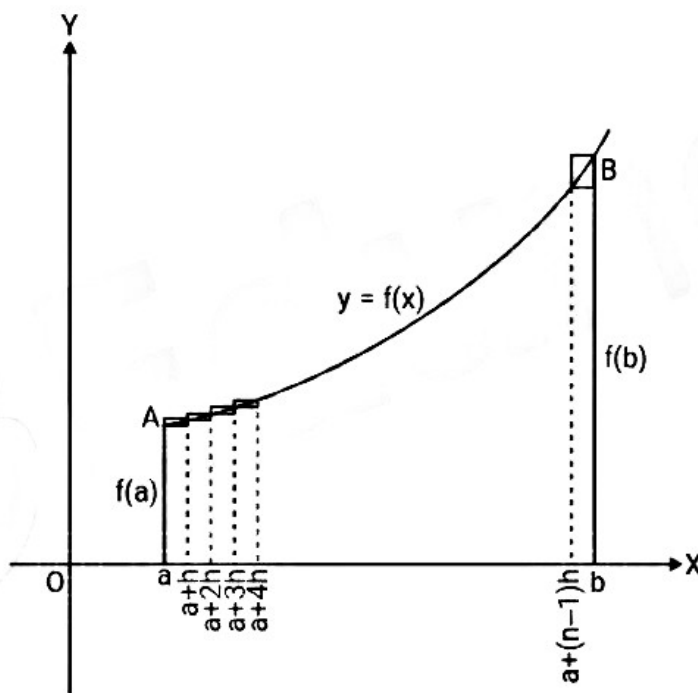
when $n \to \infty$ then $h \to 0$

$\therefore$ Area of aABb $= \int_a^b f(x)\,dx$

$$= \underset{h \to 0}{\text{Lim}} \{ hf(a) + hf(a+h) + hf(a+2h) + \dots + hf(a+\overline{n-1}h) \}$$

$$= \underset{h \to 0}{\text{Lim}} \, h \{ f(a) + f(a+h) + f(a+2h) + \dots + f(a+\overline{n-1}h) \}$$

$$= \underset{h \to 0}{\text{Lim}} \, h \sum_{r=0}^{n-1} f(a+rh)$$

Hence $\int_a^b f(x)\,dx = \underset{h \to 0}{\text{Lim}} \, h \sum_{r=0}^{n-1} f(a+rh)$

Some Important Results to Remember

(i) $\displaystyle\sum_{r=1}^{n} r = \Sigma n = \frac{n(n+1)}{2}$

(ii) $\displaystyle\sum_{r=1}^{n} a = \Sigma a = na$

(iii) $\displaystyle\sum_{r=1}^{n} r^2 = \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$

(iv) $\displaystyle\sum_{r=1}^{n} r^3 = \Sigma n^3 = (\Sigma n)^2 = \left\{\frac{n(n+1)}{2}\right\}^2 = \frac{n^2(n+1)^2}{4}$

(v) In A.P., sum of n terms, $S_n = \dfrac{n}{2}\{2a + (n-1)d\} = \dfrac{n}{2}(a+l)$

(vi) In G.P., sum of n terms, $S_n = \begin{cases} \dfrac{a(r^n-1)}{(r-1)}, & r>1 \\[3mm] \dfrac{a(1-r^n)}{(1-r)}, & r<1 \end{cases}$

(vii) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha+(n-1)\beta)$

$= \dfrac{\sin(n\beta/2)}{\sin(\beta/2)} \sin\left(\dfrac{\text{1st angle} + \text{last angle}}{2}\right)$

(viii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha+(n-1)\beta)$

$= \dfrac{\sin(n\beta/2)}{\sin(\beta/2)} \cos\left(\dfrac{\text{1st angle} + \text{last angle}}{2}\right)$

(ix) $1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \dfrac{1}{5^2} - \ldots\infty = \dfrac{\pi^2}{12}$

(x) $1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \dfrac{1}{5^2} + \ldots\infty = \dfrac{\pi^2}{6}$

(xi) $1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\infty = \dfrac{\pi^2}{8}$

(xii) $\dfrac{1}{2^2} + \dfrac{1}{4^2} + \dfrac{1}{6^2} + \ldots\infty = \dfrac{\pi^2}{24}$

(xiii) $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$, $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

(xiv) $\cos\theta + i\sin\theta = e^{i\theta}$, $\cos\theta - i\sin\theta = e^{-i\theta}$

Example : Prove that $\int_a^b x^2\, dx = \dfrac{1}{3}(b^3 - a^3)$ by first principle rule.

Solution. Here $f(x) = x^2$, $nh = b - a$ from the definition,

$$\int_a^b x^2\, dx = \lim_{h \to 0} h \sum_{r=0}^{n-1} (a + rh)^2$$

$$= \lim_{h \to 0} h \sum_{r=0}^{n-1} (a^2 + 2ahr + h^2 r^2)$$

$$= \lim_{h \to 0} h \left(\sum_{r=0}^{n-1} a^2 + 2ah \sum_{r=0}^{n-1} r + h^2 \sum_{r=0}^{n-1} r^2 \right)$$

$$= \lim_{h \to 0} h \left\{ a^2 n + 2ah \frac{(n-1)n}{2} + \frac{h^2 (n-1)\, n\, (2n-1)}{6} \right\}$$

$$= \lim_{h \to 0} \left\{ a^2 (nh) + a(nh)\, (nh - h) + \frac{(nh - h)\, (nh)\, (2nh - h)}{6} \right\}$$

$$= \lim_{h \to 0} \left\{ a^2 (b - a) + a(b-a)(b - a - h) + \frac{(b - a - h)(b - a) \times (2(b-a) - h)}{6} \right\}$$

$$= a^2 (b - a) + a(b - a)^2 + \frac{(b-a)^3}{3}$$

$$= \frac{(b - a)}{3} \{ b^2 + ab + a^2 \}$$

$$= \frac{1}{3} \{ b^3 - a^3 \}$$

Example : From the definition of a definite integral as the limit of sum evaluate $\int_a^b e^x\, dx$.

Solution. Here $f(x) = e^x$, $nh = b - a$ from the definition

$$\int_a^b e^x\, dx = \lim_{h \to 0} h \sum_{r=0}^{n-1} e^{(a + rh)}$$

$$= \lim_{h \to 0} h e^a \sum_{r=0}^{n-1} e^{rh}$$

$$= \lim_{h \to 0} h e^a \{ 1 + e^h + e^{2h} + \ldots + e^{(n-1)h} \}$$

$$= \lim_{h \to 0} h e^a \left\{ \frac{1((e^h)^n - 1)}{e^h - 1} \right\}$$

$$= e^a \lim_{h \to 0} h \left\{ \frac{e^{nh} - 1}{e^h - 1} \right\}$$

$$= e^a (e^{b - a} - 1) \lim_{h \to 0} \frac{h}{e^h - 1}$$

$$= (e^b - e^a) \cdot 1$$

$$= e^b - e^a$$

Summation of series by Integration

Let f(x) be a continuous function defined on the closed interval [a, b], then

$$\lim_{n \to \infty} \sum_{r=0}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x) \, dx$$

Rule : Here we proceed as follows :

(i) Express the given series in the form $\Sigma \frac{1}{n} f\left(\frac{r}{n}\right)$

(ii) Then the limit is its sum when $n \to \infty$

$$= \lim_{n \to \infty} \Sigma \frac{1}{n} f\left(\frac{r}{n}\right)$$

Replace $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx and $\lim_{n \to \infty} \Sigma$ the sign of $\int$.

(iii) The lower and upper limits of integration will be the values of $\frac{r}{n}$ for the first and last term (or the limit of these values) respectively.

when $r = 0 \Rightarrow x = \frac{r}{n} = 0$

$$r = n - 1 \Rightarrow x = \frac{n-1}{n} = 1 - \frac{1}{n} = 1 - 0 = 1$$

Note : In $\prod_{r=0}^{n-1} f(a + rh) = \sum_{r=0}^{n-1} \ln f(a + rh)$

Example : Find the limit, when $n \to \infty$ of

$$\frac{1}{\sqrt{(2n - 1^2)}} + \frac{1}{\sqrt{(4n - 2^2)}} + \frac{1}{\sqrt{(6n - 3^2)}} + \ldots + \frac{1}{n}$$

Solution. Let $P = \lim_{n \to \infty} \left\{ \frac{1}{\sqrt{(2n - 1^2)}} + \frac{1}{\sqrt{(4n - 2^2)}} + \frac{1}{\sqrt{(6n - 3^2)}} + \ldots + \frac{1}{n} \right\}$

$$= \lim_{n \to \infty} \left\{ \frac{1}{\sqrt{(1(2n) - 1^2)}} + \frac{1}{\sqrt{(2(2n) - 2^2)}} + \frac{1}{\sqrt{(3(2n) - 3^2)}} + \ldots + \frac{1}{\sqrt{n(2n) - n^2}} \right\}$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{\sqrt{(2nr - r^2)}}$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \frac{1}{\sqrt{2\left(\frac{r}{n}\right) - \left(\frac{r}{n}\right)^2}}$$

let $r/n = x$

at $r = 1$ and as $n \to \infty$, we get $x \to 0$

at $r = n$ and as $n \to \infty$, we get $x \to 1$

Put $x = t^2$ $\qquad \therefore \qquad dx = 2t \, dt$

when $x = 0 \Rightarrow t = 0$

$x = 1 \Rightarrow t = 1$

$\therefore \quad P = \int_0^1 \frac{2t\,dt}{t\sqrt{(2-t^2)}} = 2\sin^{-1}\left(\frac{t}{\sqrt{2}}\right)\Big|_0^1$

$= 2\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - 0 = 2\frac{\pi}{4} = \frac{\pi}{2}$

Hence $P = \pi/2$.

Example : Find the limit, when $n \to \infty$ of

$$\frac{\sqrt{n}}{(3+4\sqrt{n})^2} + \frac{\sqrt{n}}{\sqrt{2}(3\sqrt{2}+4\sqrt{n})^2} + \frac{\sqrt{n}}{\sqrt{3}(3\sqrt{3}+4\sqrt{n})^2} + ... + \frac{\sqrt{n}}{\sqrt{n}(3\sqrt{n}+4\sqrt{n})^2}$$

Solution. Let $P = \underset{n\to\infty}{\text{Lim}}\left\{\frac{\sqrt{n}}{(3+4\sqrt{n})^2} + \frac{\sqrt{n}}{\sqrt{2}(3\sqrt{2}+4\sqrt{n})^2} + \frac{\sqrt{n}}{\sqrt{3}(3\sqrt{3}+4\sqrt{n})^2} + ... + \frac{\sqrt{n}}{\sqrt{n}(3\sqrt{n}+4\sqrt{n})^2}\right\}$

$P = \underset{n\to\infty}{\text{Lim}}\left\{\frac{\sqrt{n}}{\sqrt{1}(3\sqrt{1}+4\sqrt{n})^2} + \frac{\sqrt{n}}{\sqrt{2}(3\sqrt{2}+4\sqrt{n})^2} + \frac{\sqrt{n}}{\sqrt{3}(3\sqrt{3}+4\sqrt{n})^2} + ... + \frac{\sqrt{n}}{\sqrt{n}(3\sqrt{n}+4\sqrt{n})^2}\right\}$

$= \underset{n\to\infty}{\text{Lim}} \sum_{r=1}^{n} \frac{\sqrt{n}}{\sqrt{r}(3\sqrt{r}+4\sqrt{n})^2}$

$= \underset{n\to\infty}{\text{Lim}} \sum_{r=1}^{n} \frac{1}{n\sqrt{\left(\frac{r}{n}\right)}\left\{3\sqrt{\left(\frac{r}{n}\right)}+4\right\}^2}$

$= \int_0^1 \frac{dx}{\sqrt{x}(3\sqrt{x}+4)^2}$

Put $\quad 3\sqrt{x} + 4 = t$

$\therefore \quad \frac{3}{2\sqrt{x}}dx = dt \Rightarrow \frac{dx}{\sqrt{x}} = \frac{2}{3}dt$

when $x = 0 \Rightarrow t = 4$

$x = 1 \Rightarrow t = 7$

$\therefore \quad P = \frac{2}{3}\int_4^7 \frac{dt}{t^2} = \frac{2}{3}\left(-\frac{1}{t}\right)\Big|_4^7$

$= -\frac{2}{3}\left\{\frac{1}{7} - \frac{1}{4}\right\}$

$= \frac{2}{3}\left\{\frac{1}{4} - \frac{1}{7}\right\} = \frac{2}{3}\cdot\frac{3}{28} = \frac{1}{14}$

Hence $P = \frac{1}{14}$

Gamma Function

The definite integral $\int_0^\infty e^{-x} x^{n-1} \, dx$ is called the **Second Eulerian Integral** and is denoted by the symbol Γn (read as Gamma n).

$$\therefore \quad \Gamma n = \int_0^\infty e^{-x} x^{n-1} \, dx$$

$$\Rightarrow \quad \Gamma(n+1) = n \, \Gamma n$$

This formula is known as recurrence formula for Gamma function.

Important Formula :

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{\Gamma\left(\dfrac{m+1}{2}\right)\Gamma\left(\dfrac{n+1}{2}\right)}{2\Gamma\left(\dfrac{m+n+2}{2}\right)}$$

where $m > -1$ and $n > -1$

(This formula is applicable only when the limit is 0 to $\pi/2$).

Note that : $\quad \Gamma(1/2) = \sqrt{\pi}$

Walli's formula

[An easy way to evaluate $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$ where $m, n \in I_+$]

we have $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$

$$= \frac{\{(m-1)(m-3) \ldots 2 \text{ or } 1\} \{(n-1)(n-3) \ldots 2 \text{ or } 1\}}{\{(m+n)(m+n-2)(m+n-4) \ldots 2 \text{ or } 1\}} p$$

where p is $\pi/2$ if m and n are both even, otherwise p = 1. In last factor in each of the three products is either 1 or 2. In case of m or n is 1, we simply write 1 as the only factor to replace its product. If any of m or n is zero provided we put 1 as the only factor in its product and we regard 0 as even.

Problem on Greatest Integral Function

Example : Evaluate : $\int_0^{1.5} [x^2] \, dx$, where [.] denotes the greatest integer function.

Solution. Analytical Method :

Value of x^2 at $x = 0$ is $0^2 = 0$ and at $x = 1.5$ is $(1.5)^2 = 2.25$

$\therefore$ integers between 0 to 2.25 are 1, 2

then $x^2 = 1$ and $x^2 = 2$ $\qquad \therefore \quad x = 1$ and $x = \sqrt{2}$

$$\int_0^{1.5} [x^2] \, dx = \int_0^1 [x^2] \, dx + \int_1^{\sqrt{2}} [x^2] \, dx + \int_{\sqrt{2}}^{1.5} [x^2] \, dx$$

$$= 0 + \int_1^{\sqrt{2}} 1 \cdot dx + \int_{\sqrt{2}}^{1.5} 2 \cdot dx$$

$$= 1 \cdot (\sqrt{2} - 1) + 2(1.5 - \sqrt{2})$$

$$= (2 - \sqrt{2})$$

Graphical Method :
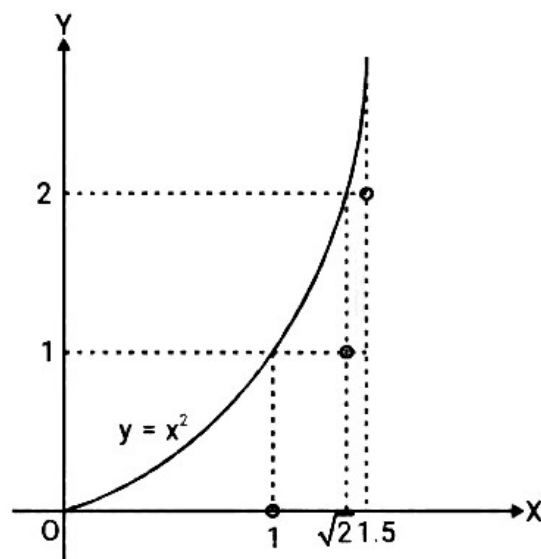
Fig.

$\int_0^{1\cdot5} [x^2]\, dx$ = Area of shaded region

$$= (\sqrt{2} - 1)\,.\,1 + (1\cdot5 - \sqrt{2})\,.\,2$$

$$= (2 - \sqrt{2})$$

Example : Prove that : $\int_0^x [t]\, dt = \dfrac{[x]\,([x] - 1)}{2} + [x]\,(x - [x])$

where [.] denotes the greatest integer function.

Solution. Analytical Method :

Let $x = n + f \;\forall\; n \in I$ and $0 \le f < 1$

$\therefore$ $[x] = n$ ...(1)

$$\int_0^x [t]\, dt = \int_0^1 [t]\, dt + \int_1^2 [t]\, dt + \int_2^3 [t]\, dt + ... + \int_n^{n+f} [t]\, dt$$

$$= 0 + 1\,.\,\int_1^2 dt + 2\int_2^3 dt + ... + n\int_n^{n+f} dt$$

$$= 1 + 2 + 3 + ... + (n - 1) + nf$$

$$= \frac{(n - 1)n}{2} + nf$$

$$= \frac{[x]\,([x] - 1)}{2} + [x]\,(x - [x]) \qquad\qquad \text{[from (1)}\}$$

Graphical Method :

Let $x = n + f \;\forall\; n \in I$ and $0 \le f < 1$

$\therefore$ $[x] = n$ ...(1)

$$\int_0^x [t]\, dt = \text{Area of shaded region}$$

$$= 0 + 1 + 2 + ... + (n - 1) + (x - n)\,n$$

Fig.
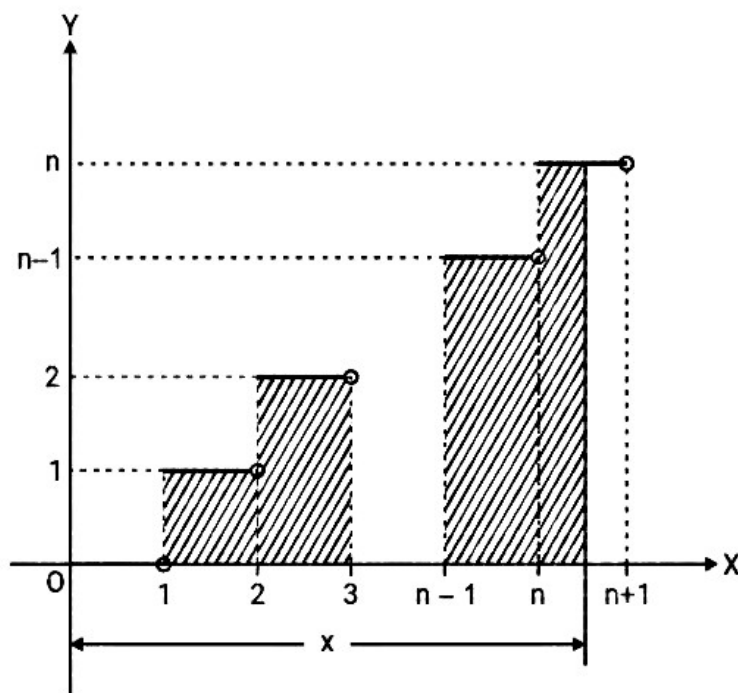
$$= \frac{(n-1)n}{2} + (x-n)n$$

$$= \frac{[x][(x)-1]}{2} + [x](x-[x])$$

Example : Evaluate : $\int_0^\pi [2 \sin x] dx$, where [.] denotes the greatest integer function.

Solution. Analytical Method :

∵ $0 \le x \le \pi$

       $0 \le 2 \sin x \le 2$

Now when $2 \sin x = 1$ $\Rightarrow$ $\sin x = 1/2$

∴ $x = \pi/6,\ 5\pi/6$

and   If $0 \le 2 \sin x < 1$, $0 < x < \dfrac{\pi}{6}$ and $\dfrac{5\pi}{6} < x < \pi$

∴ $[2 \sin x] = 0$ and   If $1 \le 2 \sin x < 2$

∴ $[2 \sin x] = 1$, $\dfrac{\pi}{6} < x < \dfrac{5\pi}{6}$

Now $\int_0^\pi [2 \sin x] dx = \int_0^{\pi/6} [2 \sin x]\, dx + \int_{\pi/6}^{5\pi/6} [2 \sin x] dx + \int_{5\pi/6}^\pi [2 \sin x] dx$

$$= 0 + 1\int_{\pi/6}^{5\pi/6} 1\, dx + 0 = \frac{5\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{3}$$

Graphical Method :

$\int_0^\pi [2 \sin x] dx$ = Area of shaded region
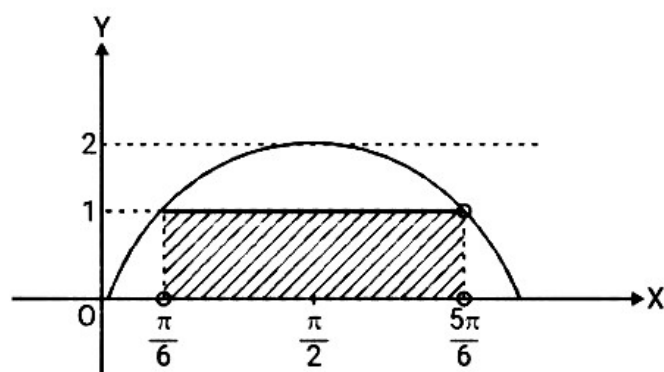
Fig.

$$= 0 + \left(\frac{5\pi}{6} - \frac{\pi}{6} \right) . 1 + 0$$

$$= \frac{2\pi}{3}$$

1. DIFFERENTIATION UNDER THE INTEGRAL SIGN

In this section, we shall prove that, under suitable conditions, 'the derivative of the integral and the integral of the derivative are equal', and consequently, 'the two repeated integrals are equal for continuous functions.'

Theorem (Leibnitz's Rule). If f is defined and continuous on the rectangle R = [a, b; c d], and if

(i) $f_x(x, y)$ exists and is continuous on the rectangle R, and

(ii) $g(x) = \int_c^d f(x,y)dy$, for $x \in [a,b]$,

then g is differentiable on [a, b], and

$$g'(x) = \int_c^d f_x(x,y)dy,$$

i.e., $\dfrac{d}{dx}\left\{\int_c^d f(x,y)dy\right\} = \int_c^d \dfrac{\partial f(x,y)}{\partial x}dy$

Corollary 1. (General Leibnitz's rule). If f satisfy the conditions of the above theorem, and if

(i) $\phi,\psi : [a, b] \to [c, d]$ are both differentiable, and

(ii) $g(x) = \int_{\phi(x)}^{\psi(x)} f(x,y)dy$, for $x \in [a,b]$,

then g is differentiable on [a, b] and

$$g'(x) = \int_{\phi(x)}^{\psi(x)} f_x(x,y)dy + f(x,\psi(x))\psi'(x) - f(x,\phi(x))\phi'(x).$$

Corollary 2. If f is continuous on R = [a, b; c, d], then

$$\int_a^b \left\{\int_c^d f(x,y)\,dy\right\}dx = \int_c^d \left\{\int_a^b f(x,y)\,dx\right\}dy.$$

Iterated Integrals (or repeated integrals)

Definition. An iterated integral is an integral of the form

$$\int_a^b dx \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy$$

where ϕ_1 or ϕ_2 or both are functions of x or constants.

This means that for each fixed x between a and b, the integral

$$F(x) = \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy$$

is evaluated, and then the integral $\int_a^b F(x)\,dx$.

$$\therefore \int_a^b F(x)\,dx = \int_a^b \left\{\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\right\}dx$$

or

$$= \int_a^b dx \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy$$

...(1)

The other repeated integral

$$\int_c^d dy \int_{\phi_1(y)}^{\phi_2(y)} f(x,y) dx \quad \text{or} \quad \int_c^d \left\{ \int_{\phi_1(y)}^{\phi_2(y)} f(x,y) dx \right\} dy \qquad \text{...(2)}$$

is defined in the same way.

Example If $| a | < 1$. show that

$$\int_0^\pi \frac{\log(1 + a\cos x)}{\cos x} dx = \pi \sin^{-1} a.$$

We write

$$f(a,x) = \frac{\log(1 + a\cos x)}{\cos x}$$

It may be seen that the integrand has only a removable discontinuity at $\pi/2$ as much as

$$\lim_{x \to \pi/2} = \frac{\log(1 + a\cos x)}{\cos x} = a$$

Also $\quad f_a(a,x) = \dfrac{1}{1 + a\cos x}$

We see that f and f_a are both continuous functions of two variables in the domain $\{(a,x) : 0 \le x \le \pi, | a | < 1\}$. If we assign to f, the value a for $x = \pi/2$.

We write

$$\phi(a) = \int_0^\pi f(a,x) dx. \qquad \text{...(1)}$$

$$\Rightarrow \quad \phi'(a) = \int_0^\pi f_a(a,x) dx = \int_0^\pi \frac{1}{1 + a\cos x} dx.$$

put $\quad \cos x = \dfrac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$

$$\int_0^\pi \frac{\sec^2(x/2)}{1 + \tan^2(x/2) + a(1 - \tan^2(x/2))} dx$$

$$\int_0^\pi \frac{\sec^2(x/2)}{(1+a) + (1-a)\tan^2(x/2)} dx$$

put $\quad \tan(x/2) = t$

diff w. r. to x we get

$$\sec^2(x/2).1/2 \, dx = dt$$

$$= \int_0^\infty \frac{2dt}{(1+a) + (1-a)t^2}$$

$$= \left[\frac{2}{\sqrt{1-a^2}} \tan^{-1}\left(\sqrt{\frac{1-a}{1+a}} t \right) \right]_0^\infty$$

$$= \left[\frac{2}{\sqrt{1-a^2}} \left(\tan^{-1}(\infty) - \tan^{-1}(0) \right) \right]$$

$$= \left[\frac{2}{\sqrt{1-a^2}} \frac{\pi}{2} \right]$$

Therefore, $\displaystyle\int_0^\pi \frac{dx}{1+a\cos x} = \frac{\pi}{\sqrt{(1-a^2)}}$

$\Rightarrow \qquad \phi'(a) = \dfrac{\pi}{\sqrt{(1-a^2)}}.$

From this we obtain

$\qquad \phi(a) = \pi \sin^{-1} a + c.$ ...(2)

where, c, is an arbitrary constant.

From (1), we have

$\qquad \phi(0) = 0.$

Putting a = 0 in (2), we obtain c = 0.

Hence $\phi(a) = \pi \sin^{-1} a.$

2. FUNDAMENTAL THEOREM OF CALCULUS

Theorem : If a function f is bounded and integrable on [a, b], then the function F defined as

$$F(x) = \int_a^x f(t)dt, \ a \le x \le b,$$

is continuous on [a, b], and furthermore, if f is continuous at a point c of [a, b], then F is derivable at c and

F'(c) = f(c).

Since f is bounded therefore ∃ a number K > 0 such that

| f(x) | < K, ∀ x ∈ [a, b].

If x_1, x_2 are two points of [a, b] such that a ≤ x_1 < x_2 ≤ b, then

$$\left| F(x_2) - F(x_1) \right| = \left| \int_{x_1}^{x_2} f(t)dt \right|$$

$$\le K(x_2 - x_1)$$

Thus for a given ε > 0, we see that

| F(x_2) − F(x_1) | < ε, if | x_2 − x_1 | < ε/K.

Hence the function F is continuous (in fact uniformly) on [a, b].

Let f be continuous at a point c of [a, b], so that for any ε > 0 there exists δ > 0 such that

| f(x) − f(c) | < ε, for | x − c | < δ

Let   c − δ < δ < c < t < c + δ

$$\therefore \quad \left| \frac{F(t)-F(s)}{t-s} - f(c) \right| = \left| \frac{1}{t-s} \int_s^t \{f(x) - f(c)\} \, dx \right| \le \frac{1}{t-s} \int_s^t |f(x) - f(c)| \, dx < \varepsilon.$$

⇒   F'(c) = f(c).

i.e., continuity of f at any point of [a, b] implies derivability of F at that point.

Note. As c is any point of [a, b], we have for all x ∈ [a, b],

F'(t) = f(t) ⇒ F = f

i.e.; continuity of f on [a, b] implies derivability of F on [a, b].

This theorem is sometimes referred to as **the First Fundamental Theorem of Integral Calculus**.

Definition : A derivable function F, if it exists such that its derivative F' is equal to a given function f, is called a primitive of f.

The above theorem shows that a sufficient condition for a function to admit of a primitive is that it is continuous. Thus every continuous function f possesses a primitive F where

$$F(x) = \int_a^x f(t)\,dt$$

Remark : We shall now show, with the help of an example, that continuity of a function is not a necessary condition for the existence of a primitive, in other words, "functions possessing primitives are not necessarily continuous".

Consider the function f on [0, 1], where

$$f(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x), & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases}$$

It has a primitive F. where

$$F(x) = \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Clearly $F'(x) = f(x)$ but $f(x)$ is not continuous at $x = 0$, i.e., $f(x)$ is not continuous on $[0, 1]$

In fact, all this amounts to saying that the derivative of a function is not necessarily continuous.

Theorem : Function f is bounded and integrable on $[a, b]$ and there exists a function F such that $F' = f$ on $[a, b]$, then

$$\int_a^b f\,dx = F(b) - F(a).$$

Since the function $F'. f$ is bounded and integrable, therefore for every given $\varepsilon > 0$, $\exists\ \delta > 0$ such that for every partition $P = \{a = x_0\ x_1...x_2......x_n = b\}$. with norm $\mu(p) < \delta$,

or

$$\left. \begin{aligned} \sum_{i-t}^{n} f(t_i)\Delta x_i - \int_a^b f\,dx &< \varepsilon \\ \operatorname*{Lim}_{(P) \to a} \sum_{i-1}^{n} f(t_i)\Delta x_i &= \int_a^b f\,dx. \end{aligned} \right\} \qquad \qquad ...(1)$$

for every choice of points t_1 in Δx_1.

Since we have freedom in the selection of points $t_1 \Delta x_1$, we choose them in a particular way as follows:

By Lagrange's mean value theorem, we have

$$F(x_t) - F(x_{t-1}) = F(t_1)\Delta x_t \quad (i = 1, 2, ..., n)$$
$$= f(t_1)\Delta x_t.$$

$$\Rightarrow \quad \sum_{i=1}^{n} f(t_1)\Delta x_t = \sum_{i=1}^{n} \{F(x_1) - F(x_{t-1})\}$$

$$= F(b) - F(a).$$

Hence from (1).

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

It is sometimes referred to as **The second Fundamental Theorem of Integral Calculus.**

Example : Show that the function $[x]$, where $[x]$ denotes the greatest integer not greater than x, is integrable in $[0, 3]$.

Since the function is bounded and has only three points of discontinuity therefore it is integrable and

$$\int_0^3 [x]\,dx = \int_0^1 [x]\,dx + \int_1^2 [x]\,dx + \int_2^3 [x]\,dx$$

$$= \int_0^1 0\,dx + \int_1^2 dx + \int_2^3 2\,dx = 3$$

1, 2 and 3 being respectively the points of discontinuity of the three integrals on the right.

Example : f is a non-negative continuous function on $[a, b]$ and $\int_a^b f\,dx = 0$. Prove that $f(x) = 0$, for all $x \in [a, b]$.

Solution. Suppose that, for some $c \in (a, b)$, $f(c) > 0$

Then, for $\varepsilon = \dfrac{1}{2} f(c) > 0$, continuity of f at c implies that, there exists a $\delta > 0$ such that

$$f(x) > \frac{1}{2} f(c), \ \forall \ x \in (c - \delta, \ c + \delta)$$

Now $\quad \displaystyle\int_a^b f(x)\,dx = \int_a^{c-\delta} f(x)\,dx + \int_{c-\delta}^{c+\delta} f(x)\,dx + \int_{c+\delta}^b f(x)\,dx$

$$\geq \int_{c-\delta}^{c+\delta} f(x)\,dx \qquad\qquad (\because \ f(x) \geq 0, \ \forall \ x \in [a, \ b])$$

$$> \frac{1}{2} f(c) \int_{c-\delta}^{c+\delta} dx = \delta f(c) > 0.$$

which is a contradiction. Thus $f(x) = 0$, $\forall \ x \in (a, b)$.

Similarly, $f(a) \geq 0$, and $f(b) \geq 0$. Hence the result follows.

3. DOUBLE INTEGRAL

The Calculation of a double integral

Equivalence of a double with repeated integrals.

Theorem. If the double integral

$$\int\int_R f(x,y)dx\,dy$$

exists where R is the rectangle [a, b; c, d]

and if also $\int_a^b f(x,y)dx$ exist $\forall\ y \in [c, d]$,

then the repeated integral

$$\int_c^d\left\{\int_a^b f(x,y)dx\right\}dy$$

exists and is equal to the double integral.

Definite Double Integral : If we have function f(x, y) then

$$\int_{y=c}^d\int_{x=\phi(y)}^{\psi(y)} f(x,y)dx\,dy \quad \text{or} \quad \int_{x=a}^b\int_{y=f(x)}^{f(x)} f(x,y)\,dy\,dx$$

If dx first then function of y is constant and if dy first then function of x is constant.

Cor. If a double integral exists, then the two repeated integrals cannot exist without being equal.

Cor. A function f is defined in [0, 1; 0, 1] as-follows :

$$f(x, y) = \frac{1}{2}, \text{ when y is rational,}$$

$$f(x, y) = x, \text{ when y is irrational;}$$

$\int_0^1\left\{\int_0^1 f(x,y)dx\right\}dy$ exists and is equal to $\frac{1}{2}$, but the double integral and the second repeated integral do not exist.

Example : Evaluate $\int\int xy(x+y)dx\,dy$ over the area between $y = x^2$ and $y = x$.

Solution. The area is bounded by the curves

$y = f_1(x) = x^2, y = f_2(x) = x$. see in fig.
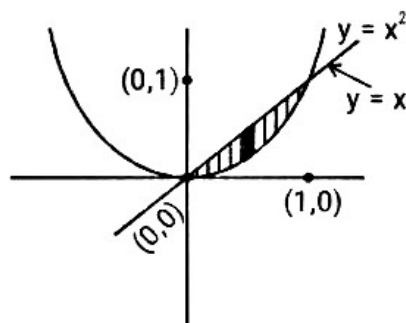
When $f_1(x) = f_2(x)$, $x^2 = x$, i.e., $x = 0$, $x = 1$, i.e., the area of integration is bounded by

$y = x^2, y = x, x = 0, x = 1$

In fig. x varies from x = 0 to x = 1 and y varies from $y = x^2$ to $y = x$

$\therefore \qquad \iint xy(x+y)\,dx\,dy = \int_0^1 \left[\int_{x^2}^x [xy(x+y)\,dy] \right] dx$

$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_{x^2}^x dx$

$= \int_0^1 \left(\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx$

$= \frac{3}{56}.$

Example : Prove that

$$\int_0^1 \left\{ \int_0^1 \frac{x-y}{(x+y)^3}\,dy \right\} dx = \frac{1}{2} \neq -\frac{1}{2} = \int_0^1 \left\{ \int_0^1 \frac{x-y}{(x+y)^3}\,dx \right\} dy$$

Solution. LHS $= \int_0^1 \left\{ \int_0^1 \frac{x-y}{(x+y)^3}\,dy \right\} dx$

$= \int_0^1 \left[\int_0^1 \left\{ \frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2} \right\} dy \right] dx$

$= \int_0^1 \left[\frac{2x.1}{-2(x+y)^2} + \frac{1}{(x+y)} \right]_0^1 dx$

$= \left[-\frac{x}{(x+1)^2} + \frac{1}{(x+1)} + \frac{x}{x^2} - \frac{1}{x} \right] dx$

$= \int_0^1 \left[-\frac{x}{(x+1)^2} + \frac{1}{(x+1)} + \frac{1}{x} - \frac{1}{x} \right] dx$

$= \int_0^1 \left[-\frac{x}{(x+1)^2} + \frac{1}{(x+1)} \right] dx$

$= \int_0^1 \left[\frac{-x+x+1}{(x+1)^2} \right] dx$

$= \int_0^1 \frac{1}{(x+1)^2}\,dx$

$= \left[\frac{-1}{(x+1)} \right]_0^1$

$= \left[-\frac{1}{2} + 1 \right] = \frac{1}{2}$

RHS $= \int_{y=0}^1 \left\{ \int_{x=0}^1 \frac{x-y}{(x+y)^3}\,dx \right\} dy$

$= \int_0^1 \left\{ \int_0^1 \left(\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \right) dx \right\} dy$

$$= \int_0^1 \left[\frac{-1}{(x+y)} + \frac{2y}{2(x+y)^2} \right]_0^1 dy$$

$$= \int_0^1 \left[\frac{-1}{1+y} + \frac{y}{(1+y)^2} + \frac{1}{y} - \frac{y}{y^2} \right] dy$$

$$= \int_0^1 \left[\frac{-1}{1+y} + \frac{y}{(1+y)^2} \right] dy$$

$$= \int_0^1 \left[\frac{-(1+y)+y}{(1+y)^2} \right] dy$$

$$= \int_0^1 \frac{-1}{(1+y)^2} dy$$

$$= \left[\frac{1}{1+y} \right]_0^1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

LHS ≠ RHS

4. CHANGE OF ORDER OF INTEGRATION

In calculus, interchange of the order of integration is a methodology that transforms iterated integrals of functions into other.

Theorem : If S is a region of Type I, then

$$\int_S f(x,y)dA = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy \right) dx$$

If S is a region of Type II, then

$$\int_S f(x,y)dA = \int_c^d \left(\int_{\phi_1(y)}^{\phi_2(y)} f(x,y)dx \right) dy$$

We should point out that, unlike the case of integrals over rectangles, there is only one order in which we can carry out the integrations. If S is a Type I region we have to integrate over y before we integrate over x and if S is a Type II region we have to carry out the integration over x before we integrate over y.

However, sometimes a region is both Type I and Type II. This does not mean that

$$\int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy \right) dx = \int_{\phi_1(x)}^{\phi_2(x)} \left(\int_a^b f(x,y)dx \right) dy$$

because the right hand side doesn't really make any sense. Rather, it means that we can prescribe the region S in two different ways

$$S = \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b, \phi_1(x) \le y \le \phi_2(x)\}$$
$$= \{(x, y) \in \mathbb{R}^2 \mid y_1(y) \le x \le y_2(y), c \le y \le d\}$$

The preceding theorem applied to this situation simple says

$$\int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy \right) dx = \int_c^d \left(\int_{\phi_1(y)}^{\phi_2(y)} f(x,y)dx \right) dy$$

In order to reverse the order of integration of an integral like

$$\int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy \right) dx$$

one therefore first has to Fig. out how to parameterize the region of integration

$$\{(x, y) \in \mathbb{R}^2 \mid a \le x \le b, \phi_1(x) \le y \le \phi_2(x)\}$$

as a Type II region; that is to say, one has to Fig. out what c, d, $\psi_1(y)$, and $\psi_2(y)$ are.

Example : Find the value of $\iint_E e^{y/x} dS$ if the domain E integration is the triangle bounded by the straight lines y = x, y = 0 and x = 1.

Solution. How to take limit of y and x, we draw a strip parallel to y axis in the fig.

Now lower end of strip is on the line y = 0 and upper end of strip is on the line y = x

So y varies from y = 0 to y = x.

For x, we pull the strip at the left end of shaded fig i.e. x = 0 and extend the strip i.e the right end of shaded region i.e. x = 1 So y varies from y = 0 to x.

To avoid integration of $e^{y/x}$ with respect to

x, we use

$$\iint_E e^{y/x} dS = \int_0^1 dx \int_0^x e^{y/x} dy$$
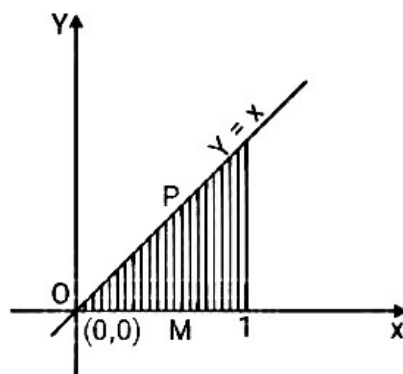
$$= \int_0^1 x(e-1)\,dx$$

$$= \frac{e-1}{2}$$

Example : Evaluate the double integral $\int_E e^{x+y}\,dx\,dy$, when E is the domain which lies between two squares of sides 2 and 4, with center at the origin and sides parallel to the axes.

Solution. Domain E is not quadratic with respect to any axes but the straight lines x = −1, x = 1 divide it into four quadratic domains, E_1, E_2, E_3, E_4.
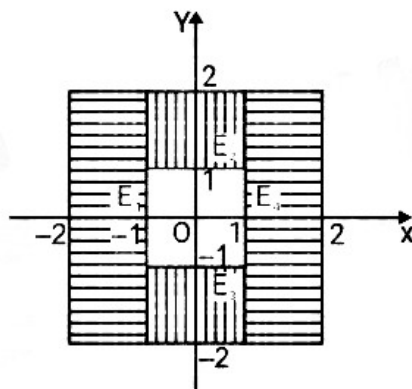
$$\iint_E e^{x+y}dxdy = \iint_{E_1} e^{x+y}dxdy + \iint_{E_2} e^{x+y}dxdy + \iint_{E_3} e^{x+y}dxdy + \iint_{E_4} e^{x+y}dxdy$$

see the region E_1, limit of x is from -2 to -1 and limit of y is from -2 to 2
Similarly find limits for other regions also.

$$\therefore \quad \iint_E e^{x+y}\,dx\,dy = \int_{-2}^{-1}dx\int_{-2}^{2}e^{x+y}\,dy + \int_{-1}^{1}dx\int_{1}^{2}e^{x+y}\,dy + \int_{-1}^{1}dx\int_{-2}^{-1}e^{x+y}\,dy + \int_{1}^{2}dx\int_{-2}^{2}e^{x+y}\,dy$$

$$= \int_{-2}^{-1}(e^{x+2} - e^{x-2})dx + \int_{-1}^{1}(e^{x+2} - e^{x+1})dx + \int_{-1}^{1}(e^{x-1} - e^{x-2})dx + \int_{1}^{2}(e^{x+2} - e^{x-2})dx$$

$$= \left[e^{x+2} - e^{x-2}\right]_{-2}^{-1} + \left[e^{x+2} - e^{x+1}\right]_{-1}^{1} + \left[e^{x-1} - e^{x-2}\right]_{-1}^{1} + \left[e^{x+2} - e^{x-2}\right]_{1}^{2}$$

$$= \left[e^{-1+2} - e^{-1-2} - e^{-2+2} + e^{-2-2}\right] + \left[e^{1+2} - e^{1+1} - e^{-1+2} + e^{-1+1}\right]$$
$$\quad + \left[e^{1-1} - e^{1-2} - e^{-1-1} + e^{-1+2}\right] + \left[e^{2+2} - e^{2-2} - e^{1+2} + e^{1-2}\right]$$

$$= \left[e^{1} - e^{-3} - e^{0} + e^{-4}\right] + \left[e^{3} - e^{2} - e^{1} + e^{0}\right]$$
$$\quad + \left[e^{0} - e^{-1} - e^{-2} + e^{3}\right] + \left[e^{4} - e^{0} - e^{3} + e^{-1}\right]$$

$$= \left[e^1 - e^{-3} - e^0 + e^{-4} \right] + \left[e^3 - e^2 - e^1 + e^0 \right]$$
$$+ \left[e^0 - e^{-1} - e^{-2} + e^{-3} \right] + \left[e^4 - e^0 - e^3 + e^{-1} \right]$$
$$= \left(e^4 + e^{-4} \right) - \left(e^2 + e^{-2} \right)$$
$$= 2 \cosh(4) - 2 \cosh(2)$$

Example : Evaluate $\iint_R f(x,y) \, dx \, dy$, over the rectangle R = [0, 1; 0, 1], where

$$f(x,y) = \begin{cases} x + y, \text{ if } x^2 < y < 2x^2 \\ 0, \quad \text{elsewhere} \end{cases}$$

Solution. For non-zero values of the function f(x, y), the domain R is divided into two domains A and B, we obtain
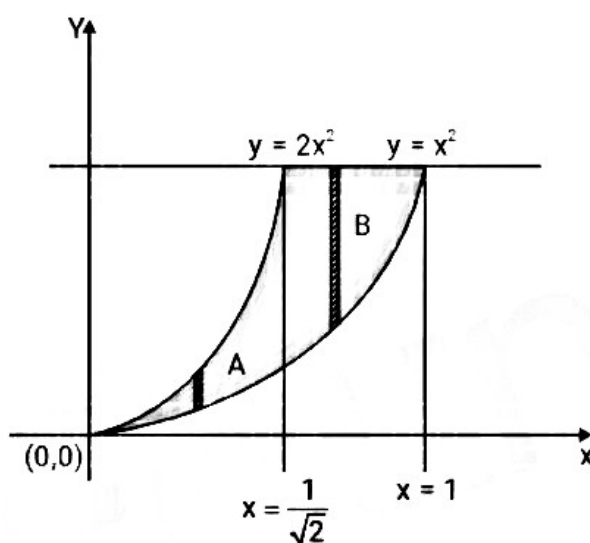
$$\iint_R f(x,y) \, dx \, dy = \iint_A (x+y) \, dx \, dy + \iint_B (x+y) \, dx \, dy$$

$$= \int_0^{1/\sqrt{2}} dx \int_{x^2}^{2x^2} (x+y) \, dy + \int_{1/\sqrt{2}}^1 dx \int_{x^2}^1 (x+y) \, dy$$

$$= \int_0^{\sqrt{2}} \left(x^3 + \frac{3}{2} x^4 \right) dx + \int_{1/\sqrt{2}}^1 \left(x + \frac{1}{2} - x^3 - \frac{x^4}{2} \right) dx$$

$$= (21 - 8\sqrt{2}) / 40.$$

Example : Evaluate $\iint_R [x + y] \, dx \, dy$, over the rectangle R = [0, 1 : 0, 2], where [x + y] denotes the greatest integer less than or equal to (x + y). We have, for (x, y) ∈ R.

$$[x + y] = \begin{cases} 0, \text{ if } 0 \le x + y < 1 \\ 1, \text{ if } 1 \le x + y < 2 \\ 2, \text{ if } 2 \le x + y < 3 \end{cases}$$
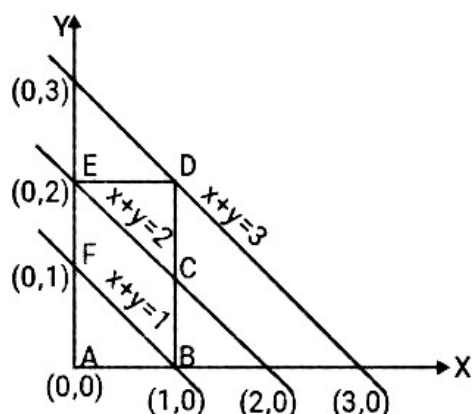
Solution. The domain of integration R (i.e. region ABDE) is divided into three domains ABF, BCEF and CDE and [x + y] = 0 in the region ABF, [x + y] = 1 in region BCEF and [x + y] = 2 in the region CDE.

Therefore, we have

$$\iint_R [x + y]\, dx\, dy = \int_0^1 dx \int_0^{1-x} [x + y]\, dy + \int_0^1 dx \int_{1-x}^{2-x} [x + y]\, dy + \int_0^1 dx \int_{2-x}^2 [x + y]\, dy$$

$$= \int_0^1 dx \int_0^{1-x} 0\, dy + \int_0^1 dx \int_{1-x}^{2-x} 1\, dy + \int_0^2 dx \int_{2-x}^2 2\, dy$$

$$= \int_0^1 [2 - x - 1 + x]\, dx + 2\int_0^2 [2 - 2 + x]\, dx$$

$$= \int_0^1 3\, dx + 2\int_0^2 x\, dx$$

$$= 3[x]_0^1 + 2\left[\frac{x^2}{2}\right]_0^2$$

$$= 3\,[1 - 0] + 2\,[2 - 0]$$
$$= 3 + 4 = 7$$

Jacobian of Transformation

f(x, y)dx dy = f(r cos θ, r sin θ) Jdr dθ

dx dy = Jdr dθ

$$J = \frac{dx\,dy}{dr\,d\theta} = \frac{\partial(x, y)}{\partial(r, \theta)}$$

(x, y) = (r, θ)
x = r cos θ
y = r sin θ
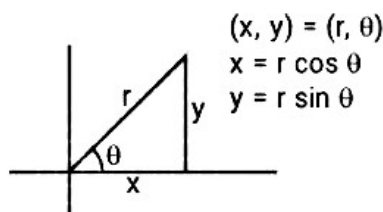
$$\therefore \qquad \frac{\partial x}{\partial r}\hat{r} + \frac{\partial x}{\partial \theta}\hat{\theta}$$

$$\frac{\partial y}{\partial r}\hat{r} + \frac{\partial y}{\partial \theta}\hat{\theta}$$

$$J = \frac{dx\,dy}{dr\,d\theta} = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\[2mm] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix}$$

which is known as Jacobian.

Jacobian of transformation form of (x, y) into (r, θ)

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\boxed{dx\,dy = J\,dr\,d\theta = r\,dr\,d\theta}$$

Change of Variables in Double Integrals

Let the functions

x = X(u, v); y = Y(u, v)

map in one-to-one manner a domain D in Cartesian coordinates (x, y) onto a domain D' in the new coordinates u, v.

Let f(x, y) = f(X(u, v), Y(u, v)) = F(u, v)

Then $\displaystyle\iint_D f(x,y)\,dx\,dy = \iint_{D'} F(u,v)\,|J|\,du\,dv$

where the Jacobian J = $\dfrac{\partial(x,y)}{\partial(u,v)}$.

In case of a transformation from a Cartesian coordinate system (x, y) to polar coordinate system (r, θ); | f | = r, and hence dx dy = r dθ dr.

Change of Variables in Triple Integrals:

Let the functions

x = X(u,v,w), y = Y(u,v,w), z = Z(u,v,w)

map in one-to-one manner, a domain D in cartesian coordinates (x,y,z) onto a domain D' in the new variables (u,v,w)

Let　f(x,y,z) = f(X(u,v,w), Y(u,v,w), Z(u,v,w))

　　　　　= F(u,v,w)

Then $\displaystyle\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{D'} F(u,v,w)\,|J|\,du\,dv\,dw$

where the Jacobian　J = $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ (matrix is same as given by eq. (1) of order 3)

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2mm] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial \omega} \end{vmatrix}$$

The following two transformations because of their frequent occurrence, deserve special mention

(i)　　　Cylindrical polar coordinates

$x = r \cos \theta$, $y = r \sin \theta$, $z = z$

Here the Jacobian $J = r$.

(ii)   Spherical polar coordinates.

$x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$

Here the Jacobian $J = r^2 \sin \theta$.

Examples : To integrate $(x^2 + y^2)$ over the circle $x^2 + y^2 = a^2$, we change to polar, $x = r \cos \theta$, $y = r \sin \theta$, so that $| J | = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$, $\iint_{x^2 + y^2 < a^2} (x^2 + y^2) dx\, dy = \int_0^{2\pi} \int_0^a r^2 . r\, dr\, d0 = \dfrac{\pi a^4}{2}$

Solution : $\iint_{x^2 + y^2 = a^2} (x^2 + y^2) dx\, dy$

$\int_{\theta=0}^{2\pi} \int_{r=0}^{a} r^2\, r\, d\theta\, dr$

$\int_0^{2\pi} \left[\dfrac{r^4}{4} \right]_0^a d\theta$ (integrating wrt r)
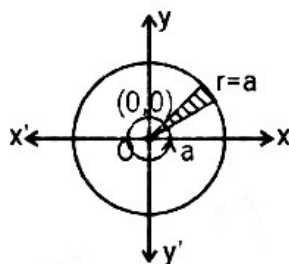
$x = r \cos \theta$
$y = r \sin \theta$
$dx\, dy = r\, d\theta\, dr$
or $dx\, dy = |J|\, d\theta\, dr$

$\int_0^{2\pi} \dfrac{a^4}{4} d\theta$

$= \dfrac{a^4}{4} \left(\int_0^{2\pi} d\theta \right) = \dfrac{a^4}{4} \left[\theta \right]_0^{2\pi} = \dfrac{a^4}{4} \left[2\pi - 0 \right] = \dfrac{a^4}{4} \times 2\pi = \dfrac{\pi a^4}{2}$

1. To evaluate $\iint \sqrt{\dfrac{a^2 b^2 - b^2 x^2 - a^2 y^2}{a^2 b^2 + b^2 x^2 + a^2 y^2}}\, dx\, dy$, over the positive quadrant of the ellipse $x^2/a^2 + y^2/b^2$

$= 1$, by putting $x = au \Rightarrow dx = a\, du$, and $y = bv \Rightarrow dy = bdv$, we have to evaluate $ab \iint \sqrt{\dfrac{1 - u^2 - v^2}{1 + u^2 + v^2}}\, du\, dv$, over the positive quadrant of the circle $u^2 + v^2 = 1$.
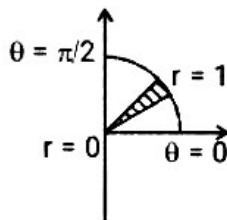
Again, changing to polars, by putting $u = r \cos \theta$, $v = r \sin \theta$, the integral becomes

$= ab \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} \sqrt{\dfrac{1 - r^2}{1 + r^2}} r\, dr\, d\theta$

$= ab \int_0^1 \sqrt{\dfrac{1 - r^2}{1 + r^2}} r\, dr \int_0^{\pi/2} d\theta$

put $1 + r^2 = \rho^2 \Rightarrow 2rdr = 2\rho d\rho$ (limit: $r = 0 \Rightarrow \rho = 1$, $r = 1 \Rightarrow \rho = \sqrt{2}$)

$$= \frac{1}{2}\pi ab \int_1^{\sqrt{2}} \frac{\sqrt{2-\rho^2}}{\rho} \rho d\rho$$

$$= \frac{1}{2}\pi ab \int_1^{\sqrt{2}} \sqrt{2-\rho^2} d\rho$$

put $\rho = \sqrt{2} \sin t$

$d\rho = \sqrt{2} \cos t \; dt$

$$= \frac{\pi ab}{2} \int_1^{\sqrt{2}} \sqrt{2-\rho^2} d\rho$$

Put $\rho = \sqrt{2} \sin t$

$d\rho = -\sqrt{2} \cos t \; dt$

$$= \frac{\pi}{2} ab \int_{\theta=\pi/4}^{\pi/2} \sqrt{2} \cos \sqrt{2} \cos t \; dt$$

$$= \frac{\pi}{2} ab \int_{\theta=\pi/4}^{\pi/2} 2 \cos^2 t \; dt$$

$$= \frac{\pi}{2} ab \int_{\theta=\pi/4}^{\pi/2} (1+\cos 2t) dt$$

$$= \frac{\pi}{2} ab \left[t + \frac{\sin 2t}{2} \right]_{\pi/4}^{\pi/2}$$

$$= \frac{\pi}{2} ab \left[\frac{\pi}{2} - \frac{\pi}{4} + \frac{\sin \pi}{2} - \frac{\sin \pi/2}{2} \right]$$

$$= \frac{\pi}{2} ab \left[\frac{\pi}{4} - \frac{1}{2} \right] \text{ Ans.}$$

Note : The transformation could be effected in one step by putting $x/a = r \cos \theta$, $y/b = r \sin \theta$, then $|J| = abr$.

2. To evaluate $\iint \{2a^2 - 2a(x+y) - (x^2+y^2)\} dx \; dy$ over the circle $x^2 + y^2 + 2a(x+y) = 2a^2$, transform the origin to $(-a, -a)$, by putting $x + a = u$, $y + a = v$, so that the integral becomes $\iint (4a^2 - u^2 - v^2) \; du \; dv$, over the circle $u^2 + v^2 = 4a^2$. Changing to polars, we get $8\pi a^4$.

$I = \iint \{2a^2 - 2a(x+y) - (x^2+y^2)\} dx \; dy$ over the

circle $x^2 + y^2 + 2a(x+y) = 2a^2$

$x^2 + 2ax + a^2 + y^2 + 2ay + a^2 = 4a^2$

$(x+a)^2 + (y+a)^2 = 4a^2$

put $x + a = u$, $y + a = v$

So the integral becomes

$$I = \iint (4a^2 - u^2 - v^2)\, du\, dv$$

$r = 2a$

$r=0$

$\theta = 0$ to 2π
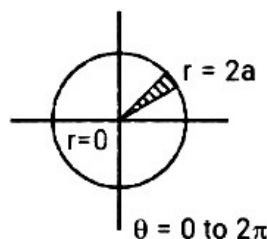
Changing into polar coordinates

$$= \int_{\theta=0}^{2\pi} \int_{x=0}^{2a} (4a^2 - r^2)r\, d\theta\, dr$$

$$= \int_0^{2\pi} \left[4a^2 \frac{r^2}{2} - \frac{r^4}{4} \right]_0^{2a} d\theta$$

$$= \left[4a^2 \cdot \frac{4a^2}{2} - \frac{16a^4}{4} \right] \int_0^{2\pi} d\theta$$

$$= 4a^4 \left[\theta \right]_0^{2\pi} = 4a^4 (2\pi)$$

$$= 8\pi a^4$$

Example : Evaluate $\iint (y-x)\, dx\, dy$, over the region E in the xy-plane bounded by the straight lines

$$y = x - 3,\ y = x + 1,\ 3y + x = 5,\ 3y + x = 7$$

Solution. It is difficult to evaluate the double integral directly; however a simple change of coordinates reduces the domain of integration into a rectangle with sides parallel to the axis.

Set $\quad y - x = u,\ 3y + x = v$

so that $\quad x = \dfrac{1}{4}(v - 3u),\ y = \dfrac{1}{4}(v + u)$

and $\quad J = \dfrac{1}{16} \begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix} = -\dfrac{1}{4},\quad \therefore |J| = -\dfrac{1}{4}$

The new domain is the rectangle R = [–3, 1; 5, 7] in uv-plane.

$$\therefore \quad \iint_E (y - x)\, dx\, dy = \iint_R u \cdot \frac{1}{4} du\, dv$$

$$= \frac{1}{4} \int_{-3}^{1} u\, du \int_5^7 dv = -2$$

Example : Evaluate the integral

$$I = \int_0^1 dx \int_0^x \sqrt{x^2 + y^2}\, dy$$

by passing on to the polar coordinates.

Solution. The integral in question is the double integral $\iint \sqrt{x^2 + y^2}\, dx\, dy$ over the region enclosed by the triangle $y = 0,\ y = x,\ x = 1$.

In polar coordinates, the equations of these lines are $\theta = 0,\ \theta = \pi/4$. $r \cos \theta = 1$, so that the domain of integration is $0 \le \theta \le \pi/4,\ 0 \le r \le \sec \theta$.

$$\therefore \qquad I = \int_0^{\pi/4} d\theta \int_0^{\sec\theta} r \cdot r \ dr \ d\theta = \frac{1}{3}\int_0^{\pi/4} \sec^3\theta \ d\theta$$

$$= \frac{1}{6}\left[\sqrt{2} + \log(1+\sqrt{2})\right]$$

Dirichlet's Theorem

The theorem states that

$$\iiint \cdots \int x_1^{m_1-1} x_2^{m_2-1} x_3^{m_3-1}, \ x_n^{m_n-1} dx_1 \ dx_2 \ dx_3 \ \dots \ dx_n$$

$$= \frac{\overline{|m_1} \ \overline{|m_2} \ \overline{|m_3} \ \dots \ \overline{|m_n}}{\overline{|(m_1 + m_2 + \dots + m_n + 1)}}$$

where the integral is extended to all +ve values of the variables x_1, x_2, ..., x_n subject to the condition $x_1 + x_2 + \dots, x_n \le 1$.

Liouville's Extension of Dirichlet's Theorem

If x, y, z all positive such that

$$h_1 \le x + y + z \le h_2$$

$$\iiint F(x+y+z) x^{\ell-1} \ y^{m-1} \ z^{n-1} \ dx \ dy \ dz$$

$$= \frac{\overline{|\ell} \ \overline{|m} \ \overline{|n}}{\overline{|\ell+m+n}} \int_{h_1}^{h_2} F(u) \ u^{\ell+m+n-1} \ du$$

Example : Evaluate

$$\iint_E x^{m-1} \ y^{n-1}(1-x-y)^{p-1} \ dx \ dy, \ m \ge 1, \ n \ge 1, p \ge 1$$

where E is the region bounded by $x = 0$, $y = 0$, $x + y = 1$.

Now $\quad \displaystyle\iint_E x^{m-1} \ y^{n-1}(1-x-y)^{p-1} \ dx \ dy$

$$= \int_0^1 dx \int_0^{1-x} x^{m-1} \ y^{n-1}(1-x-y)^{p-1} \ dy$$

For this type of integrals, two sets of substitutions are possible, which give an integral with constant limits. We discuss both these here.

First method. Put $x + y = u$, $x = uv$ so that

$$y = u(1-v)$$

and $\quad J = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u$

The Jacobian vanishes when $u = 0$, i.e., when $x = 0 = y$. The origin of the xy-plane corresponds to the whole line $u = 0$ of the uv-plane, so that the correspondence ceases to be one-to-one. To exclude the origin of the xy-plane, we cut out the region along a line $x = h$ parallel to the y-axis and consider the integral on the remaining domain E_1, bounded by the lines

$$v = 1, \quad u = 1, \quad uv = h$$
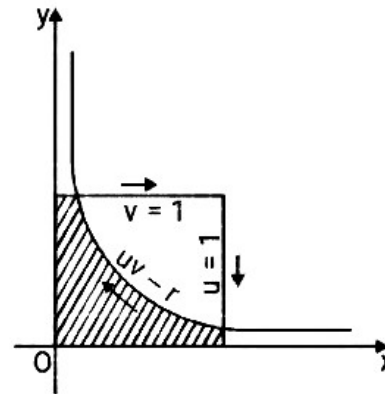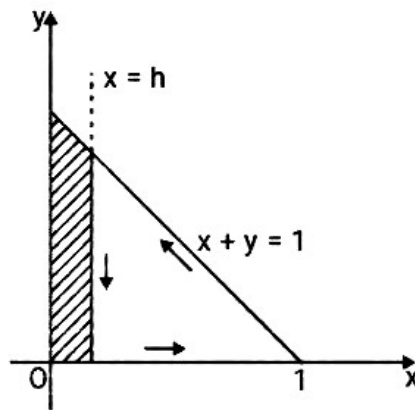
However, in the limit when $h \to 0$, this new region degenerates into the square bounded by

$$v = 1, \quad u = 1, \quad v = 0, \quad u = h$$

Thus $\iint_{\delta} x^{m-1} y^{n-1} (1 - x - y)^{p-1} \, dx \, dy$

$$= \int_0^1 \int_0^1 u^{m-1} u^{m-1} (1 - v)^{n-1} (1 - u)^{p-1} u \, du \, dv$$

$$= \int_0^1 u^{m+n-1} (1 - u)^{p-1} \, du \int_0^1 v^{m-1} (1 - v)^{n-1} \, dv$$

$$= \beta(m + n, \, p) . \beta(m, \, n)$$

$$= \frac{\Gamma(m + n) \, \Gamma(p)}{\Gamma(m + n + p)} \frac{\Gamma(m) \, \Gamma(n)}{\Gamma(m + n)} = \frac{\Gamma(m) \, \Gamma(n) \, \Gamma(p)}{\Gamma(m + n + p)}$$

Second method. Put $x = u$, $y = (1 - u)v$, so that

$$1 - x - y = 1 - u - (1 - u)v = (1 - u)(1 - v)$$

and $J = \begin{vmatrix} 1 & 0 \\ -v & 1 - v \end{vmatrix} = 1 - v$

which vanishes for $u = 1$, so that the point $(1, 0)$ in the xy-plane corresponds to the whole line $u = 1$ in the uv-plane. However, proceeding as in the first method, the new region becomes the square.

$$u = 0, \quad v = 0, \quad u = 1, \quad v = 1$$

Thus $\iint_E x^{m-1} y^{n-1} (1 - x - y)^{p-1} \, dx \, dy$

$$= \int_0^1 \int_0^1 u^{m-1} v^{n-1} (1 - u)^{p+n-1} (1 - v)^{p-1} \, du \, dv$$

$$= \int_0^1 u^{m-1} (1 - u)^{p+n-1} \, du \int_0^1 v^{n-1} (1 - v)^{p-1} \, dv$$

$$= \beta(m, \, p + n) \cdot \pi(n, \, p)$$

$$= \frac{\Gamma(m) \, \Gamma(p + n)}{\Gamma(m + n + p)} \cdot \frac{\Gamma(n) \, \Gamma(p)}{\Gamma(p + n)} = \frac{\Gamma(m) \, \Gamma(n) \, \Gamma(p)}{\Gamma(m + n + p)}$$

Example : Evaluate $\iint_E \sin\left(\dfrac{x - y}{x + y}\right) dxdy$, where E is the region bounded by the co-ordinate axes and $x + y = 1$ in the first quadrant.

Solution. Taking $x - y = u$, $x + y = v$, so that $x = \dfrac{1}{2}(u+v)$, $y = \dfrac{1}{2}(v-u)$, and the Jacobian is $\dfrac{1}{2}$.

$\therefore \qquad \iint_B \sin\left(\dfrac{x-y}{x+y}\right)dxdy = \iint_{Euy} \sin\left(\dfrac{u}{v}\right)\dfrac{1}{2}dudv$ ...(1)
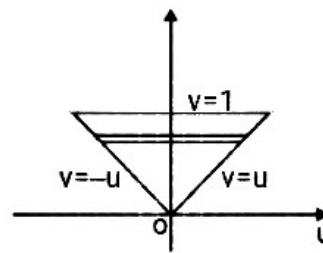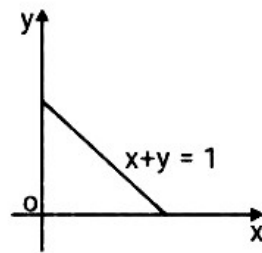
Now $\qquad \iint_{Euy} \sin\left(\dfrac{u}{v}\right)\dfrac{1}{2}dudv = \dfrac{1}{2}\int_0^1 dv \int_{-v}^{v} \sin\dfrac{u}{v}du$

$= \dfrac{1}{2}\int_0^1 v\{-\cos 1 + \cos(-1)\}dv = 0$ ...(2)

Hence, from (1) and (2), the required integral is zero.

5. TRIPLE INTEGRALS

Let f(x, y, z) be a function of three independent variables x, y and z defined at every point of a three-dimensional region V. Divide the region V into n elementary volumes δV_1, δV_2, ..., δV_n and let (x_r, y_r, z_r) be any point inside the rth sub-division δV_r. Find the sum

$$\sum_{r=1}^{n} f(x_r, y_r, z_r)\delta V_r.$$

Then $\quad \iiint_V f(x,y,z)dV = \lim_{\substack{n\to\infty \\ \delta V_r\to 0}} \sum_{r=1}^{n} f(x_r, y_r, z_r)\delta V_r.$

To extend definition of repeated integrals for triple integrals, consider a function f(x, y, z) and keep x and y constant and integrate with respect to z between limits in general depending upon x and y. This would reduce f(x, y, z) to a function of x and y only. Thus, let

$$\phi(x,y) = \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz.$$

Then in φ (x, y) we can keep x constant and integrate with respect to y between limits in general depending upon x this leads to a function of x alone, say

$$\psi(x) = \int_{y_1(x)}^{y_2(x)} \phi(x,y)\,dy$$

Finally ψ(x) is integrated with respect to x assuming that the limits for x are from a to b. Thus

$$\iiint_V f(x,y,z)dV = \int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_a^b \left[\int_{y_1(x)}^{y_2(x)} \left\{ \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz \right\} dy \right] dx.$$

If we put f(x, y, z) = 1, then the volume

$$V = \iiint_V dV = \int_b^a \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} dx\,dy\,dz.$$

Use of definite triple integral in finding volume of 3D shape :

If we have a 3D-shape in which two very-very closed point (P, Q) are taken as diagonal of a cuboid.
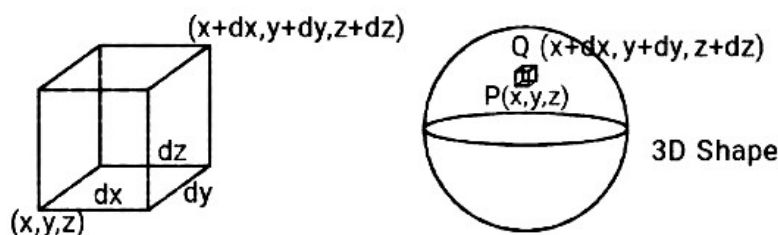
Then volume of this elementary cuboid inside that 3D shape

dV = dx dy dz

$V = \iiint dx\, dy\, dz$ taken over region

$$V = \int_{z=a}^{b} \int_{y=\psi_1(z)}^{\psi_2(z)} \int_{x=\phi_1(y,z)}^{\phi_2(y,z)} dx\, dy\, dz$$

Example : Evaluate the integral

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dx\, dy\, dz}{(x+y+z+1)^3}.$$

Solution. Let

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dx\, dy\, dz}{(x+y+z+1)^3}$$

$$= \int_0^1 \int_0^{1-x} \left[-\frac{1}{2} \frac{1}{(x+y+z+1)^2} \right]_0^{1-x-y} dx\, dy$$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-x} \left[\frac{1}{4} - \frac{1}{(x+y+1)^2} \right] dx\, dy$$

$$= -\frac{1}{2} \int_0^1 \left[\frac{1}{4} y + \frac{1}{x+y+1} \right]_0^{1-x} dx$$

$$= -\frac{1}{2} \int_0^1 \left[\frac{1}{4}(1-x) + \frac{1}{2} - \frac{1}{x+1} \right] dx$$

$$= \frac{1}{2} \int_0^1 \left(\frac{3}{4} - \frac{1}{4} x - \frac{1}{x+1} \right) dx$$

$$= -\frac{1}{2} \left[\frac{3}{4} x - \frac{1}{8} x^2 - \log(x+1) \right]_0^1$$

$$= \frac{1}{2} \left(\log 2 - \frac{5}{8} \right).$$

Example : Evaluate $\iiint_V (x^2 + y^2 + z^2)\, dx\, dy\, dz$ where V is the volume of the cube bounded by the coordinate planes and the planes $x = y = z = a$.

Solution. Here a column parallel to z-axis is bounded by the planes z = 0 and z = a.

Here the region S above which the volume V stands is the region ist he xy-plane bounded by the lines x = 0, x = a, y = 0, y = a.

Hence the given integral

$$= \int_0^a \int_0^a \int_0^a (x^2 + y^2 + z^2)\, dx\, dy\, dz$$

$$= \int_0^a \int_0^a \left[x^2 z + y^2 z + \frac{z^3}{3} \right]_0^a dx\, dy$$

$$= \int_0^a \int_0^a \left(x^2 a + y^2 a + \frac{1}{3} a^3 \right) dx\, dy$$

Fig.
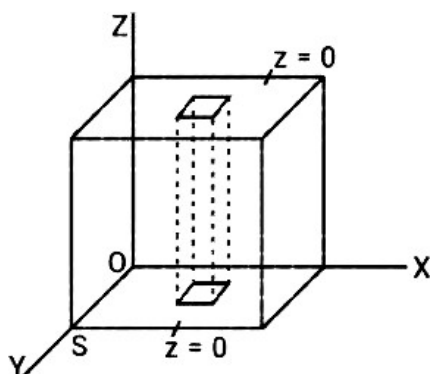
$$= \int_0^a \left[x^2 ay + \frac{1}{3}y^3 a + \frac{1}{3}a^3 y \right]_0^a dx$$

$$= \int_0^a \left(x^2 a^2 + \frac{1}{3}a^4 + \frac{1}{3}a^4 \right) dx$$

$$= \left[\frac{1}{3}x^3 a^3 + \frac{1}{3}a^4 x + \frac{1}{3}a^4 x \right]_0^a = \frac{1}{3}\left[2a^5 + a^6 \right].$$

Example : $\iiint_V (2x + y) dx \, dy \, dz$, where V is the closed region bounded by the cylinder z = 4 – x² and the planes x = 0, y = 0, y = 2 and z = 0.

Solution. Here a column parallel to z-axis is bounded by the plane z = 0 and the surface z = 4 – x² of the cylinder.

This cylinder z = 4 – x² meets the z-axis, x = 0, y = 0, at (0, 0, 4) and the x-axis, y = 0, z = 0 at (2, 0, 0) in the given region.

Therefore, it is evident that the limits of integration for z are from 0 to 4 – x², for y from 0 to 2 and for x from 0 to 2.

Here the given integral

Fig.
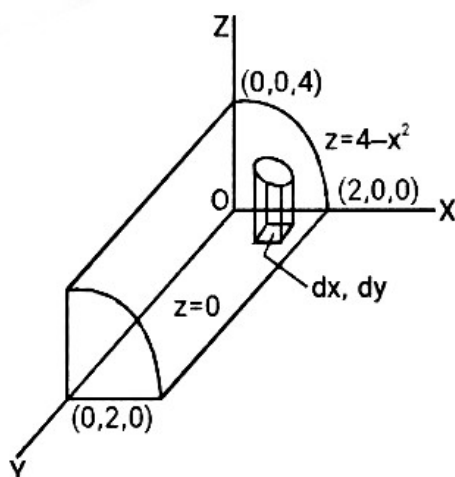
$$= \int_{x=0}^{2} \int_{y=0}^{2} \int_{z=0}^{4-x^2} (2x + y) \, dx \, dy \, dz$$

$$= \int_{x=0}^{2} \int_{y=0}^{2} (2x + y) \, [z]_0^{4-x^2} \, dx \, dy$$

$$= \int\limits_{x=0}^{2} \int\limits_{y=0}^{2} (2x + y)(4 - x^2) \, dx \, dy$$

$$= \int\limits_{x=0}^{2} \int\limits_{y=0}^{2} [8x - 2x^3 + (4 - x^2)\,y] \, dx \, dy$$

$$= \int\limits_{x=0}^{2} \left[8xy - 2x^3 y + \frac{1}{2}(4 - x^2)y^2 \right]_0^2 dx$$

$$= \int\limits_{0}^{2} [16x - 4x^3 + 2(4 - x^2)] \, dx$$

$$= \left[8x^2 - x^4 + 8x - \frac{2}{3}x^3 \right]_0^2$$

$$= \left(31 - 16 + 16 - \frac{16}{3} \right) = \frac{80}{3}.$$

Example : Evaluate

$$\iiint_E z^2 \, dx \, dy \, dz$$

taken over the region common to the surfaces

$$x^2 + y^2 + z^2 = a^2, \text{ and } x^2 + y^2 = ax$$
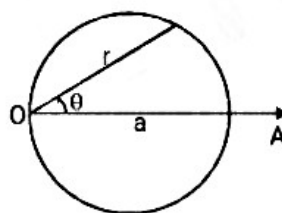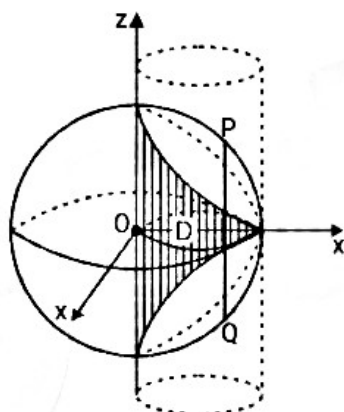
Solution. The region is bounded, above and below by the surfaces

$$z = \sqrt{a^2 - x^2 - y^2} \text{ and } z = -\sqrt{a^2 - x^2 - y^2}$$

and its projection on the xy-plane is the circular domain $D \equiv x^2 + y^2 \leq ax$.

$$\therefore \quad \iiint_E z^2 \, dx \, dy \, dz = \iint_D dx \, dy \int_{-\sqrt{(a^2 - x^2 - y^2)}}^{\sqrt{(a^2 - x^2 - y^2)}} z^2 \, dz$$

$$= \frac{2}{3} \iint_D (a^2 - x^2 - y^2)^{3/2} \, dx \, dy$$

Changing to polars, the region D becomes the circle, r = a cos θ
$0 \leq \theta \leq \pi, \, 0 \leq r \leq a \cos \theta$

$$= \frac{2}{3} \int_0^\pi d\theta \int_0^{a \cos \theta} (a^2 - r^2)^{3/2} r \, dr$$

$$= \frac{2}{15} a^5 \int_0^{\pi/2} (1 - \sin^5 \theta) d\theta = \frac{2a^5(15\pi - 16)}{225}$$

Example : Evaluate

$$I = \iiint_E (y^2z^2 + z^2x^2 + x^2y^2)\,dx\,dy\,dz$$

taken over the domain bounded by the cylinder $x^2 + y^2 = 2ax$, and the cone $z^2 = k^2(x^2 + y^2)$.

Solution. The domain E is bounded above and below by the surface

$$z = k\sqrt{x^2 + y^2}$$

and $z = -k\sqrt{x^2 + y^2}$

and its projection on the xy-plane is the circular domain D, $x^2 + y^2 \leq 2ax$.

$\therefore$ $\displaystyle I = \iint_D dx\,dy \times \int_{-k\sqrt{(x^2+y^2)}}^{k\sqrt{(x^2+y^2)}} \{(x^2 + y^2)z^2 + x^2y^2\}\,dz$

$$= 2\iint_D \left[\frac{1}{3}(x^2 + y^2)^2 k^2 + x^2y^2\right] k\sqrt{x^2 + y^2}\,dx\,dy$$

Changing to polars,

$$= \frac{4k}{3}\int_0^{\pi/2} d\theta \int_0^{2a\cos\theta}(k^2 + 3\cos^2\theta\sin^2\theta)r^6\,dr$$

$$= \frac{512}{21}ka^7 \int_0^{\pi/2}(k^2\cos^7\theta + 3\cos^9\theta\sin^2\theta)d\theta$$

$$= \frac{8192}{735}ka^7\left(k^2 + \frac{8}{33}\right)$$
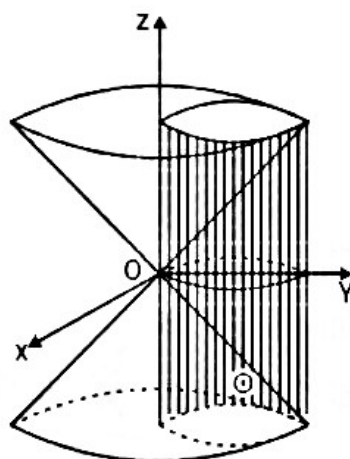
Change of Variables in A Triple Integral

Let f be continuous in a domain E bounded by a surface S in xyz-space.

Also let

$$x = x\,(X, Y, Z),\ y = y\,(X, Y, Z);\ z = z\,(X, Y, Z).$$

define three functions of three variables X, Y, Z, defined in a domain E_1 bounded by a surface S_1 in XYZ-space.

We suppose that these three functions with values

$$x\,(X, Y, Z),\ y(X, Y, Z),\ z(X, Y, Z)$$

(i)   possess continuous first order partial derivatives at each point of E_1 and S_1.

(ii)   transform E_1 into E and S_1 into S.

(iii)   the transformation is one-one.

(iv)   the Jacobian

$$\frac{\partial(x, y, z)}{\partial(X, Y, Z)}$$

does not change sign at any point of E_1 even though it may vanish at some points of S_1.

It will then be proved that

$$\iiint_E f(x, y, z)\,dx\,dy\,dz$$

$$= \iiint_{E_1} f(X, Y, Z)\left|\frac{\partial(x, y, z)}{\partial(X, Y, Z)}\right|dX\,dY\,dZ,$$

where x, y, z have to be replaced by their values in terms of X, Y, Z in f(x, y, z) in the integral on the right.

Example : Show that

$$I = \iiint x^{1-x} y^{m-1} z^{n-1} (1 - x - y - z)^{p-1} dx \, dy \, dz, \qquad (l, m, n, p \geq 1)$$

taken over the tetrahedron bounded by the planes, x = 0, y = 0, z = 0,

$$x + y + z = 1 \text{ is } \frac{\Gamma(l) \, \Gamma(m) \, \Gamma(n) \, \Gamma(p)}{\Gamma(l + m + n + p)}$$

The given integral is same as

$$\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} x^{l-1} y^{m-1} z^{n-1} (1 - x - y - z)^{p-1} dz$$

First Method. Let us put x + y + z = u, x + y = uv, x = uvw, i.e., std. substitution

$$x = uvw, \quad y = uv(1 - w), \quad z = u(1 - v)$$

It may be seen that when x, y, z are positive and x + y + z ≤ 1, then each of u, v, w lie between 0 and 1 and conversely. So the given region is fully described when 0 ≤ 1, 0 ≤ v ≤ 1, 0 ≤ w ≤ 1.

Also, them

$$|J| = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \left| -u^2 v \right| = u^2 v$$

∴
$$I = \int_0^1 u^{l+m+n-1} (1 - u)^{p-1} \, du \int_0^1 v^{l+m-1} (1 - v)^{n-1} dv$$

$$\int_0^1 w^{l-1} (1 - w)^{m-1} \, dw$$

$$= \beta(l + m + n, p) \cdot \beta(l + m, n) \cdot \beta(l, m) = \frac{\Gamma(l) \, \Gamma(m) \, \Gamma(n) \, \Gamma(p)}{\Gamma(l + m + n + p)}$$

Second Method. Put x = u, y = (1 − u)v, z = (1 − u)(1 − v)w so that
$$1 - x - y - z = (1 - u)(1 - v)(1 - w)$$

and
$$|J| = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = (1 - u)^2 (1 - v)$$

∴
$$I = \int_0^1 u^{l-1} (1 - u)^{m+n+p-1} \, du \int_0^1 v^{l-1} (1 - v)^{n+p-1} \, dv \int_0^1 w^{n-1} (1 - w)^{p-1} \, dw$$

$$= \beta(l, m + n + p) \cdot \beta(m, n + p) \cdot \beta(n, p) = \frac{\Gamma(l) \, \Gamma(m) \, \Gamma(n) \, \Gamma(p)}{\Gamma(l + m + n + p)}$$

Example : Evaluate

$$I = \iiint \sqrt{\left(a^2 b^2 c^2 - b^2 c^2 x^2 - c^2 a^2 y^2 - a^2 b^2 z^2 \right)} dx \, dy \, dz$$

taken throughout the domain

$$\left\{ (x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$$

Change the variables x, y, z to X, Y, Z where
$$x = aX, \quad y = bY, \quad z = cZ,$$
so that $\partial(x, y, z)/\partial(X, Y, Z) = abc$.

Thus $\quad I = a^2 b^2 c^2 \iint \sqrt{\left(1 - X^2 - Y^2 - Z^2\right)}\ dX\ dY\ dZ$

taken throughout $X^2 + Y^2 + Z^2 \le 1$.

Changing X, Y, Z to polar co-ordinates r, θ, φ so that

$\qquad X = r \sin\theta \cos\varphi,\ Y = r \sin\theta \sin\varphi,\ Z = r \cos\theta,$

We have, since

$\qquad \partial(X,\ Y,\ Z)/\partial(r,\ \theta,\ \varphi) = r^2 \sin\theta,$

$\qquad I = a^2 b^2 c^2 \iiint \sqrt{\left(1 - r^2\right)}\ r^2\ \sin\theta\ dr\ d\theta\ d\varphi.$

It is easily seen that to describe the whole region.

$\qquad X^2 + Y^2 + Z^2 \le 1.$

r varies from 0 to 1; θ varies from 0 to π ; φ varies from 0 to 2π.

Thus (r, θ, φ) varies in the rectangular parallelepiped.

$\qquad\qquad [0,\ 1;\ 0,\ \pi;\ 0,\ 2\pi]$

$\therefore \qquad I = a^2 b^2 c^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\ d\theta \int_0^1 \sqrt{\left(1 - r^2\right)}\ r^2\ dr = \dfrac{1}{4} a^2 b^2 c^2 \pi^2$

6. SURFACE AREA

Definition : The area of the surface

$$x = f(u, v), \quad y = g(u, v), \quad z = h(u, v); \quad (u, v) \in D$$

is the double integral

$$\iint_D \sqrt{\left\{ \left[\frac{\partial(y,z)}{\partial(u,v)} \right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)} \right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)} \right]^2 \right\}} \, du \, dv,$$

assuming that the functions f, g, h possess continuous first order partial derivatives in D and at no point of D.

$$\left[\frac{\partial(y,z)}{\partial(u,v)} \right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)} \right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)} \right]^2 = 0.$$

Note : Here, we do not propose to derive the expression as given above by basing the derivation on some elementary notion of surface area. Let the definition as given appear very arbitrary, we outline some considerations to call forth reader's faith in the same.

I. Let the surface be plane. We take

$$x = u, \quad y = v, \quad z = 0$$

where (u, v) ranges over a domain D in the XY-plane. In fact, the surface coincides with D in the present case.

We have

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \quad \frac{\partial(y,z)}{\partial(u,v)} = 0, \quad \frac{\partial(z,x)}{\partial(u,v)} = 0,$$

$$\Rightarrow \quad \iint_D \sqrt{\left\{ \left[\frac{\partial(y,z)}{\partial(u,v)} \right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)} \right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)} \right]^2 \right\}} \, du \, dv$$

$$= \iint_D du \, dv = \text{area of D}.$$

Thus the definition of surface area as now given agrees with that of areas of plane regions
.

II. Consider the surface given by

$$z = h(x, y).$$

We take it as given by

$$z = u, \quad y = v, \quad z = h(u, v)$$

where $(u, v) = (x, y) \in D \subset R^3$.

We have

$$\frac{\partial(y,z)}{\partial(u,v)} = -\frac{\partial z}{\partial u} = -\frac{\partial z}{\partial x}$$

$$\frac{\partial(z,x)}{\partial(u,v)} = -\frac{\partial z}{\partial v} = -\frac{\partial z}{\partial y}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = 1.$$

$\therefore \quad$ surface area $= \iint_D \left[1 + \left(\dfrac{\partial z}{\partial x} \right)^2 + \left(\dfrac{\partial z}{\partial y} \right)^2 \right]^{\frac{1}{2}} dx \, dy.$ \hspace{1cm} ...(1)

The reader acquainted with the elements of Differential Geometry knows that

$$-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1$$

are the direction ratios of the normal to the surface at any point (x, y, z), so that if we suppose $\delta\sigma$ to be an element of surface lying on the tangent plane at the point, the projection area δA of $\delta\sigma$ on the XY-plane is given by

$$\delta A = \frac{\delta\sigma}{\sqrt{\left[1 + \left(\dfrac{\partial z}{\partial x} \right)^2 + \left(\dfrac{\partial z}{\partial y} \right)^2 \right]}}$$

$$\Rightarrow \quad \delta\sigma = \sqrt{\left[1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right]} \, \delta A$$

$$= \sqrt{\left[1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right]} \, \delta x \, \delta y.$$

This suggests, on summation, the truth of (1).

Note. The reader may easily show that

$$\left[\frac{\partial(y, z)}{\partial(u, v)} \right]^2 + \left[\frac{\partial(z, x)}{\partial(u, v)} \right]^2 + \left[\frac{\partial(x, y)}{\partial(u, v)} \right]^2 = EG - F^2.$$

where $E = \left(\dfrac{\partial x}{\partial u} \right)^2 + \left(\dfrac{\partial y}{\partial u} \right)^2 + \left(\dfrac{\partial z}{\partial u} \right)^2,$

$$F = \frac{\partial x}{\partial u} \cdot \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \cdot \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \cdot \frac{\partial z}{\partial v}$$

$$G = \left(\frac{\partial x}{\partial v} \right)^2 + \left(\frac{\partial y}{\partial v} \right)^2 + \left(\frac{\partial z}{\partial v} \right)^2$$

Smooth surface.

Definition: A surface given by

$\quad x = f(u, v), y = g(u, v), z = h(u, v); (u, v) \in D$

is said to be smooth, if f, g, h possess continuous first order partial derivatives at each point of D and at no point there of

$$\frac{\partial(y, z)}{\partial(u, v)}, \frac{\partial(z, x)}{\partial(u, v)}, \frac{\partial(x, y)}{\partial(u, v)}$$

vanish simultaneously.

Example : The area of the surface of the paraboloid $2z = x^2 + y^2$ which lies between the planes $z = 0, z = 2$ is _____

Here $\dfrac{\partial z}{\partial x} = x, \dfrac{\partial z}{\partial y} = y$

$\therefore$ $S = \iint \sqrt{\left(1 + x^2 + y^2\right)}\, dx\, dy$

The projection on the plane z = 2 is $x^2 + y^2 = 4$ or r = 2 which is a circle between θ = 0 and θ = 2π (changing to polars by putting x = r cos θ, y = r sin θ). Hence

$$S = \int_0^{2\pi}\int_0^2 \sqrt{\left(1 + r^2\right)}\, r\, d\theta\, dr$$

$$= \frac{1}{2}\int_0^{2\pi}\left[\frac{\left(4 + 4r^2\right)^{3/2}}{2.4.3/2}\right]_0^2 d\theta$$

$$= \frac{1.2^3}{12 \times 2}\left[5\sqrt{5} - 1\right]\left[\theta\right]_0^{2\pi}$$

$$= \frac{2\pi}{3}\left[5\sqrt{5} - 1\right]$$

Example : Find the area of the surface az = xy that lies inside the cylinder $(x^2 + y^2)^2 = 2a^2xy$.

Sol. We have az = xy.

$\therefore$ $\dfrac{\partial z}{\partial x} = \dfrac{y}{a}, \dfrac{\partial z}{\partial y} = \dfrac{x}{a}$

Surface area = $\sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2} = \sqrt{\dfrac{x^2 + y^2 + a^2}{a^2}}$

$= \dfrac{1}{a}\sqrt{\left(a^2 + r^2\right)}$, changing into polars.$(x^2 + y^2 = r^2,\ x = r\cos\theta,\ y = r\sin\theta)$

Also $(x^2 + y^2)^2 = 2a^2xy$ becomes $r^2 = a^2 \sin 2\theta$

curve is a symmetric about origin and see the loop of the curve lies between θ = 0 to $\pi/2$.
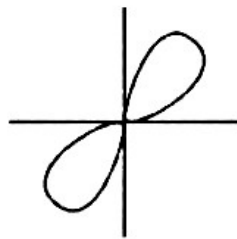
$\therefore$ $S = 2\displaystyle\int_0^{\pi/2}\int_0^{a\sqrt{(\sin 2\theta)}} \dfrac{\sqrt{\left(a^2 + r^2\right)}}{a}\, r\, d\theta\, dr$

$$= \frac{2}{a}\int_0^{\pi/2}\left[\frac{\left(a^2 + r^2\right)^{3/2}}{2 \times 3/2}\right]_0^{a\sqrt{(\sin 2\theta)}} d\theta$$

$$= \frac{2}{3}a^2\int_0^{\pi/2}\left\{\left(1 + \sin 2\theta\right)^{3/2} - 1\right\} d\theta$$

$$= \frac{2}{3}a^2\int_0^{\pi/2}\left[\left(\sin\theta + \cos\theta\right)^3 - 1\right]d\theta$$

$$S = \frac{2}{3}a^2 I \qquad \qquad \text{...(i)}$$

where $\quad I = \int_0^{\pi/2}\left[(\sin\theta + \cos\theta)^3 - 1\right]d\theta$

$$I = \int_0^{\pi/2}\left[(\sin\theta + \cos\theta)^3 - 1\right]d\theta$$

$$I = \int_0^{\pi/2}\left[\sin^3\theta + \cos^3\theta + 3\sin^2\theta\cos\theta + 3\sin\theta\cos^2\theta - 1\right]d\theta$$

$$I = \int_0^{\pi/2}\left[\sin^3\theta + \cos^3\theta + 3(1-\cos^2\theta)\cos\theta + 3\sin\theta(1-\sin^2\theta) - 1\right]d\theta$$

$$I = \int_0^{\pi/2}\left[\sin^3\theta + \cos^3\theta + 3\cos\theta - 3\cos^3\theta + 3\sin\theta - 3\sin^3\theta - 1\right]d\theta$$

$$I = \int_0^{\pi/2}\left[3\cos\theta + 3\sin\theta - 2\cos^3\theta - 2\sin^3\theta - 1\right]d\theta$$

$$I = \left[3\sin\theta - 3\cos\theta\right]_0^{\pi/2} - 2\left(\frac{2}{3} + \frac{2}{3}\right) - \left(\theta\right)_0^{\pi/2}$$

$$I = 3 + 3 + 2.\frac{4}{3} - \frac{\pi}{2} = 6 - \frac{8}{3} - \frac{\pi}{2}$$

$$I = \frac{18 - 8}{3} - \frac{\pi}{2}$$

$$I = \frac{10}{3} - \frac{\pi}{2}$$

$$I = \frac{(20 - 3\pi)}{6} \qquad \qquad \text{...(ii)}$$

By equation (i),

$$S = \quad S = \frac{2}{3}a^2 \times \frac{(20-3\pi)}{6} = \frac{a^2}{9}(20 - 3\pi)$$

Example : Compute the surface area S of the sphere

$$x^2 + y^2 + z^2 = a^2$$

The surface area of the sphere is twice the surface area of the upper half-sphere $z = \sqrt{a^2 - x^2 - y^2}$.

Now $\quad \dfrac{\partial z}{\partial x} = -\dfrac{x}{\sqrt{a^2 - x^2 - y^2}}, \quad \dfrac{\partial z}{\partial y} = -\dfrac{y}{\sqrt{a^2 - x^2 - y^2}}$

so that $\quad \sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2} = \dfrac{a}{\sqrt{a^2 - x^2 - y^2}}$

The domain of integration is the circle $x^2 + y^2 = a^2$ on the xy-plane.

Thus by formula (1), we have

$$\frac{1}{2}S = \int_{-a}^{a} dx \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} \frac{a}{\sqrt{a^2-x^2-y^2}} \, dy$$

On passing to polars, we have

$$S = 2\int_{0}^{2\pi} \left[\int_{0}^{a} \frac{a}{\sqrt{a^2-r^2}} r \, dr \right] d\theta = 4\pi a^2$$

Example : Find the area of that part of the surface of the cylinder $x^2 + y^2 = a^2$ which is cut out by the cylinder $x^2 + z^2 = a^2$.
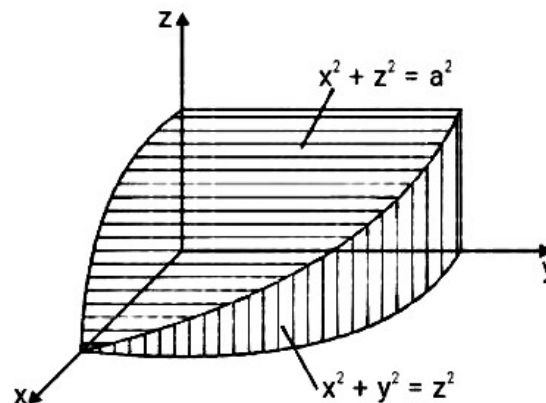
The Fig. shows $\frac{1}{8}$th of the desired surface

The equation of the surface has the form

$$y = \sqrt{a^2 - x^2}$$

so that $\dfrac{\partial y}{\partial x} = -\dfrac{x}{\sqrt{a^2-x^2}}, \dfrac{\partial y}{\partial z} = 0$

$$\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2} = \frac{a}{\sqrt{a^2-x^2}}$$

The domain of integration is a quarter circle $x^2 + z^2 = a^2$, $x \geq 0$, $z \geq 0$, on the xz-plane. Thus by formula (3), we have

$$\frac{1}{8}S = \int_{0}^{a} dx \int_{0}^{\sqrt{(a^2-x^2)}} \frac{a}{\sqrt{a^2-x^2}} \, dz = a^2$$

$\therefore$ $S = 8a^2$.

7. EVALUATION OF VOLUMES

By Triple Integration

The triple integral

$$\iiint_E dx\, dy\, dz,$$

carried throughout a region E in space of three dimensions gives the volume of E.

By Double Integration

Let C be the boundary of a region E of the xy-plane and let a cylinder be constructed by lines through the points of C parallel to z-axis. Then the volume of the cylinder enclosed between the surfaces

$$z = \varphi(x,\, y),\ z = \psi(x,\, y),\ [\varphi(x,\, y) \geq \psi(x,\, y)]$$

is $\iint\left(\int_{\psi(x,y)}^{\varphi(x,y)} dz\right) dx\, dy$, and can be easily seen, given by the double integral

$$\iint_E (\varphi(x,y) - \psi(x,y))\, dx\, dy,$$

Example. The volume enclosed by the surfaces $x^2 + y^2 = cz$, $x^2 + y^2 = 2ax$, $z = 0$.

Sol. The limits of z are 0 to $(x^2 + y^2)/c$. The limits of y are from $-\sqrt{(2ax - x^2)}$ to $\sqrt{(2ax - x^2)}$ and the limits of x are from 0 to 2a.

$\therefore$ The value

$$V = 2\int_0^{2a} \int_0^{\sqrt{(2a-x^2)}} \int_0^{(x^2-y^2)/c} dx\, dy\, dz$$

$$= 2\int_0^{2a} \int_0^{\sqrt{(2ax-x^2)}} \frac{x^2 - y^2}{c} dx\, dy$$

$$= \frac{2}{c}\int_0^{2a}\left[x^2\sqrt{(2ax - x^2)} + \frac{1}{3}(2ax - x^2)^{3/2} \right] dx$$

$$= \frac{2}{c}\int_0^{2a}\left[x^2\sqrt{\{a^2 - (a - x)^2\}} + \frac{1}{3}\{a^2 - (a - x)^2\}^{3/2} \right] dx$$

Put $a - x = a \sin\theta$

$$= -\frac{2}{c}\left[\int_{\pi/2}^{-\pi/2} (a - a\sin\theta)^2 a^2 \cos^2\theta\, d\theta + \frac{1}{3}\int_{\pi/2}^{\pi/2} a^4 \cos^4\theta\, d\theta \right]$$

$$= \frac{2a^4}{c}\int_{-\pi/2}^{\pi/2}(\cos^2\theta + \cos^2\theta\sin^2\theta - 2\cos^2\theta\sin\theta - \frac{1}{3}\cos^4\theta)$$

$$= \frac{3\pi a^4}{2c}.$$

Example : The sphere $x^2 + y^2 + z^2 = a^2$ is pierced by the cylinder $(x^2 + y^2)^2 = a^2(x^2 - y^2)$; prove that the volume of the sphere that lies inside the cylinder is $\dfrac{8}{3}\left(\dfrac{\pi}{4} + \dfrac{5}{3} - \dfrac{4\sqrt{2}}{3}\right)a^3$.

Here limits of z are from $-\sqrt{(a^2 - x^2 - y^2)}$ to $\sqrt{(a^2 - x^2 - y^2)}$ and therefore,

the volume $= \iiint dx\, dy\, dz$

$$= 2\iint \sqrt{(a^2 - x^2 - y^2)}\, dx\, dy$$

Now the equation of the cylinder is

$(x^2 + y^2)^2 = a^2(x^2 - y^2)$

Putting $x = r\cos\theta$, $y = r\sin\theta$, we get

$r^2 = a^2 \cos 2\theta$.

∴ The limits of r are from $-a\sqrt{(\cos 2\theta)}$ to $a\sqrt{(\cos 2\theta)}$ and limits of θ are from $-\pi/4$ to $\pi/4$.

∴ The required volume

$$= 2\int_{-\pi/4}^{\pi/4}\int_{-a\sqrt{(\cos 2\theta)}}^{a\sqrt{(\cos 2\theta)}} \sqrt{(a^2 - r^2)}\, r\, d\theta\, dr$$

$$= 8\int_0^{\pi/4} \frac{1}{2}\left[-\frac{(a^2 - r^2)^{3/2}}{3/2} \right]_0^{a\sqrt{(\cos 2\theta)}} d\theta$$

$$= \frac{8}{3}a^3 \int_0^{\pi/4}\left(1 - 2^{3/2}\sin^3\theta\right)d\theta$$

$$= \frac{8}{3}a^3 \int_0^{\pi/4}\left\{1 - 2^{3/2}\left(1 - \cos^2\theta\right)\sin\theta\right\}d\theta$$

$$= \frac{8}{3}a^3\left[\frac{\pi}{4} - 2^{3/2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{3 \times 2\sqrt{2}} + 1 + \frac{1}{3}\right)\right]$$

$$= \frac{8}{3}a^3\left[\frac{\pi}{4} + \frac{5}{3} - \frac{4\sqrt{2}}{3}\right].$$

1. SCALAR AND VECTOR

A VECTOR is a quantity having both magnitude and direction, such as displacement, velocity, force, and acceleration.

Graphically a vector is represented by an arrow OP (Fig.I) defining the direction, the magnitude of the vector being indicated by the length of the arrow. The tail end O of the arrow is called the *origin* or *initial point* of the vector, and the head P is called the *terminal point* or *terminus*.

Analytically, a vector is represented by a letter with an arrow over it, as A in Fig. (I) and its magnitude is denoted by $|\vec{A}|$ or A. In printed works, bold faced type, such as **A**, is used to indicate the vector $\vec{A}$ while |A| or A indicates its magnitude. The vector OP is also indicated as $\overline{OP}$ or OP; in such case we shall denote its magnitude by $\overline{OP}, |\overline{OP}|$, or |OP|.
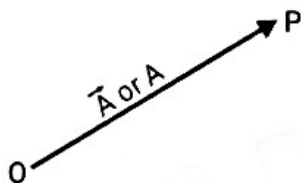
Fig. (I)

A SCALAR is a quantity having magnitude but no direction, e.g. mass, length, time, temperature, and any real number. Scalars are indicated by letters in ordinary type as in elementary algebra. Operations with scalars follows the same rules as in elementary algebra.

Vector Algebra

The operations of addition, subtraction and multiplication of scalars of vectors.

1. The vectors A and B are *equal* if they have the same magnitude and direction regardless of the position of their initial points. Thus A = B in Fig. (II)

2. A vector having direction opposite to that of vector A but having the same magnitude is denoted by – A Fig. (III)
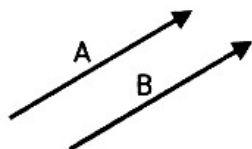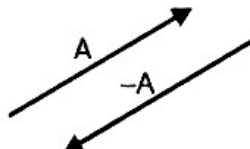
Fig. (II)   **Fig. (III)**

3. The *sum* or resultant of vectors A and B is a vector C formed by placing the initial point of B on the terminal point of A and then joining the initial point of A to the terminal point of B. This sum is written A + B, i.e. C = A + B.

4. The *difference* of vectors A and B, represented by A – B, is that vector C which added to B yields vector A. Equivalently, A – B can be defined as the sum A + (–B).

5. The *product* of a vector A by a scalar m is a vector mA with magnitude |m| times the magnitude of A and with direction the same as or opposite to that of A, according as m is positive or negative. If m = 0, mA is the null vector.

Laws of Vector Algebra

If A, B and C are vectors and m and n are scalars, then

1.	A + B = B + A	Commutative Law of Addition
2.	A + (B + C) = (A + B) + C	Associative Law of Addition
3.	mA = Am	Commutative Law of Multiplication
4.	m(nA) = (mn)A	Associative Law for Multiplication
5.	(m + n)A = mA + nA	Distributive Law
6.	m(A + B) = mA + mB	Distributive Law

A UNIT VECTOR is a vector having unit magnitude, If A is a vector with magnitude A ≠ 0, then $\dfrac{\vec{A}}{|A|}$ is a unit vector having the same direction as A.

The Rectangular Unit Vector i, j, k

An important set of unit vectors are those having the directions of the positive x, y, and z axes of a three dimensional rectangular coordinate system, and are denoted respectively by i, j, and k Fig. (IV).
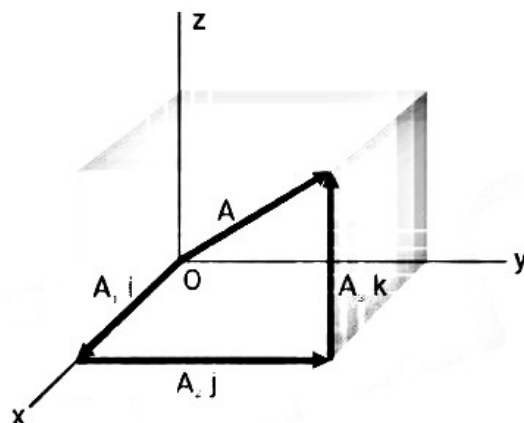
Fig. (IV)

Components of a Vector

Any vector A in 3 dimensions with initial point at the origin O of a rectangular coordinate system Fig. (IV). Let (A₁, A₂, A₃) be the rectangular coordinates of the terminal point of vector A with initial point at O. The vectors A₁i, A₂j, and A₃k are called the *rectangular component vectors* or simply *component vectors* of A is the x, y and z directions respectively. A₁, A₂ and A₃ are called the *rectangular components* or simply *components* of A in the x, y and z directions respectively.

The sum or resultant of A₂i, A₂j and A₃k is the vector A so that we can write

A = A₁i + A₂j + A₃k

The magnitude of A is

$$A = |A| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

In particular, the *position vector* or *radius vector* r from O to the point (x, y, z) is written as

r = xi + yj + zk

and the magnitude

$$r = |r| = \sqrt{x^2 + y^2 + z^2} \ .$$

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Scalar and Vector Point Functions Related to Field

(a) Scalar Field : If to each point (x, y, z) of a region R in space there corresponds a number or scalar ϕn (x, y, z), then n is called a scalar function of position or scalar point function and we say that a scalar field n has been defined in R.

A scalar field which is independent of time is called a stationary or steady state scalar field.

Examples :

1.   n(x, y, z) = $x^3y - z^2$ defines a scalar field.

2.   The temperature at any point within or on the earth's surface at a certain time defines a scalar field.

(b) Vector Field : If to each point (x, y, z) of a region R in space there corresponds a vector $\vec{A}$ (x, y, z) then $\vec{A}$ is called a vector function of position or vector point function and we say that a vector field $\vec{A}$ has been defined in R.

A vector field which is independent of time is called stationary or steady state vector field.

Examples :

1. $\vec{A} = xy\hat{i} + y^2z\hat{j} + z^3x\hat{k}$ is a vector field.

2.   If the velocity at any point (x, y, z) within a moving fluid is known at a certain time, it is said to be a vector field.

2. VECTOR FUNCTION OF A SINGLE SCALAR VARIABLE

Scalar Function

Let ¡ be the set of real numbers and $D \subset$ ¡.

If corresponding to every $t \in D$, $\phi(t)$ is a unique scalar quantity, then $\phi(t)$ is called a **scalar function** of the variable t.

Vector Function : If corresponding to every $t \in D$, f(t) is a unique vector quantity, then f(t) is called a **vector function** of the variable t. If for every single value of t, f(t) has a **unique** value, then this is called a **single valued** vector function.

Every vector can be expressed as a **Linear Combination [LC]** of given three non-coplanar vectors.

Therefore f(t) can be expressed in the following **decompose** form

$$f(t) = f_1(t)i + f_2(t)j + f_3(t)k,$$

where $f_1(t)$, $f_2(t)$, $f_3(t)$ are scalar functions of scalar variable t.

When a vector function r = f(t) is expressed in the component form, then that represents the **position vectors** of different points in space for different values of t(which is called the *parameter*). As t changes, the end point of r = f(t) determines a continuous curve which is called **space curve.**

Example : Let P be the position of a moving particle at any instant t along a curve whose position vector is r wrt the origin O. As the particle moves, the vector r also changes. Therefore r can be taken as the vector function of the time t.

Thus velocity and acceleration of a moving point are **vector function** of time t.

Conversely : A curve can always be represented in terms of position vector r of a point situated on it, where r is a function of scalar variable t (parameter).

Example : Standard Vector Equation of a Circle

We know that the parametric co-ordinates of any point on the circle with centre at the origin and radius a are

$$x = a \cos t, \quad y = a \sin t \qquad \qquad ...(1)$$

Therefore, substituting from (1) in r = xi + yj + zk, the required standard vector equation is

$$r = (a \cos t)i + (a \sin t)j + (0)k,$$

where a is a constant and t is a scalar.

Example : Standard Vector Equation of Parabola :

$$r = (at^2)i + (2at)j + (0)k, \quad \text{where a is a constant.}$$

Example : Standard Vector Equation of Ellipse :

$$r = (a \cos t)i + (b \sin t)j + (0)k,$$

where a and b are constant.

Limit of a Vector Function

Definition

A vector function f(t) is said to tend to a vector l called its limit, when t tends to t_0, if given any number ε (however small) > 0, there corresponds a number $\delta > 0$ such that

$$0 < |t - t_0| \le \delta \Rightarrow |f(t) - l| < \varepsilon$$

Thus if the limit of f(t) is l when t tends to t_0, then this can be symbolically expressed in the form:

$$\lim_{t \to t_0} f(t) = l$$

Some Theorems on Limits

Following are some of the theorem on limits of vector functions :

If $f(t)$ and $g(t)$ are two vector functions of a scalar variable t and $\phi(t)$ is scalar function of t, then:

(i) $\quad \lim_{t \to t_0} \left[f(t) \pm g(t) \right] = \lim_{t \to t_0} f(t) \pm \lim_{t \to t_0} g(t)$

(ii) $\quad \lim_{t \to t_0} \left[f(x) \cdot g(t) \right] = \left[\lim_{t \to t_0} f(t) \right] \cdot \left[\lim_{t \to t_0} g(t) \right]$

(iii) $\quad \lim_{t \to t_0} \left[f(x) \times g(t) \right] = \lim_{t \to t_0} f(t) \times \lim_{t \to t_0} g(t)$

(iv) $\quad \lim_{t \to t_0} \phi(t)\, f(t) = \left[\lim_{t \to t_0} \phi(t) \right] \left[\lim_{t \to t_0} f(t) \right]$

(v) $\quad \lim_{t \to t_0} \left| f(t) \right| = \left| \lim_{t \to t_0} f(t) \right|$

Continuity of a Vector Function

Definition

Any vector function $f(t)$ is said to be continuous at $t = t_0$, if corresponding to every positive quantity ε (however small), there exists a positive quantity δ such that

$$|t - t_0| < \delta \Rightarrow |f(t) - f(t_0)| < \varepsilon.$$

Clearly, the vector function

$f(t)$ is continuous at

$$t = t_0 \Leftrightarrow \lim_{t \to t_0} f(t) = f(t_0).$$

Derivative of a Vector Function

Let $r = f(t)$ be a single valued continuous function of scalar variable t. Let δt be the arbitrary small increase in t and δr be the corresponding increase in r, i.e. $\delta r = f(t + \delta t) - f(t)$

If the limiting value of the ratio $\delta r/\delta t$ when $\delta t \to 0$ exist, then this is called the **Derivative** or **Differential coefficient** of r wrt t and is expressed by dr/dt.

Therefore, $\dfrac{dr}{dt} = \lim_{\delta t \to 0} \dfrac{\delta r}{\delta t} = \lim_{\delta t \to 0} \dfrac{f(t + \delta t) - f(t)}{\delta t}$

For any vector $r = f(t)$ if dr/dt exist, then this called **differentiable or derivable vector.**

Geometrical interpretation of the derivative of a Vector Function :

Let $r = f(t)$ be a single valued continuous function of scalar variable t which represent a curve in the space. Let r and $r + \delta r$ be the values of this function corresponding to t and $t + \delta t$ which represent the position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ of two neighbouring points P and Q.

Therefore $r = f(t) = \overrightarrow{OP}$

and $\quad r + \delta r = f(t + \delta t) = \overrightarrow{OQ}$

$\quad \delta r = (r + \delta r)$

$\quad = \overrightarrow{OQ} - \overrightarrow{OP} = \overrightarrow{PQ}$
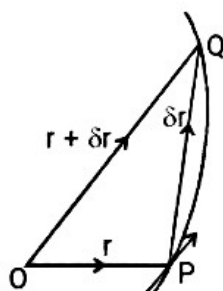
Fig.

Hence $\dfrac{\delta r}{\delta t}$ is a vector parallel to the vector $\overrightarrow{PQ}$ whose magnitude is $1/\delta t$ times the magnitude of

δr. When $\delta t \to 0$, then $\delta r \to 0$ and in that case the direction of the vector $\overrightarrow{PQ}$ is along the direction of

the tangent at the point P. Therefore the limiting value of $\dfrac{\delta r}{\delta t}$ when $\delta t \to 0$ i.e. $\dfrac{dr}{dt}$ is a vector whose

direction is along the tangent to $r = f(t)$ at the point P.

Successive Derivatives

If dr/dt is again differentiable wrt t, then its derivative is denoted by d²r/dt² which is called the **second derivative** of **r**. Similarly, if d³r/dt³ exist, then this is called the **third derivative** of **r** etc. These are all called the **successive derivatives** of **r** and denoted by $\dot{r}, \ddot{r}, \dddot{r},...$ respectively.

Derivative of a vector in terms of its components

Let $r = xi + yj + zk$, where x, y, z are any functions of the scalar variable t. Corresponding to the small arbitrary increase δt in t, there are increments δx, δy, δz and x, y, z and r respectively. Then

$$r + \delta r = (x + \delta x)i + (y + \delta y)j + (z + \delta z)k$$
$$\Rightarrow \quad \delta r = \{(xi + yj + zk) + (\delta xi + \delta yj + \delta zk)\} - r$$
$$\Rightarrow \quad \delta r = \delta xi + \delta yj + \delta zk$$
$$\Rightarrow \quad \frac{\delta r}{\delta t} = \frac{\delta x}{\delta t}i + \frac{\delta y}{\delta t}j + \frac{\delta z}{\delta t}k$$
$$\Rightarrow \quad \lim_{\delta t \to 0}\frac{\delta r}{\delta t} = \lim_{\delta t \to 0}\frac{\delta x}{\delta t}i + \lim_{\delta t \to 0}\frac{\delta y}{\delta t}j + \lim_{\delta t \to 0}\frac{\delta z}{\delta t}k$$
$$\Rightarrow \quad \frac{dr}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$$

Constant Vector : Definition : A vector having constant magnitude and direction is called a constant vector.

Note : This may be clear that any vector is not a constant vector if only either magnitude or the direction is constant.

Derivative of a constant vector

Let $r = c$ be a constant vector function of the scalar variable t. Therefore c is a constant vector.
$$\therefore \quad r + \delta r = c \Rightarrow dr = c - c = 0$$
$$\Rightarrow \quad \frac{\delta r}{\delta t} = \frac{0}{\delta t} = 0$$
$$\Rightarrow \quad \lim_{\delta t \to 0}\frac{\delta r}{\delta t} = \lim_{\delta t \to 0} 0 = 0 \Rightarrow \frac{dr}{dt} = 0$$

Therefore the derivative of a constant vector is zero vector (0).

Differentiation formulae for vectors

If a, b and c are differentiable vector functions of the same scalar variable t and ϕ is a scalar function of t, then :

(i) Derivative of the Sum and Difference of two vector functions

$$\frac{d}{dt}(a \pm b) = \frac{da}{dt} \pm \frac{db}{dt}$$

Proof. $\dfrac{d}{dt}(a + b) = \lim\limits_{\delta t \to 0} \dfrac{\{(a + \delta a) + (b + \delta b) - (a + b)\}}{\delta t}$

$= \lim\limits_{\delta t \to 0} \dfrac{\delta a + \delta b}{\delta t} = \lim\limits_{\delta t \to 0} \dfrac{\delta a}{\delta t} + \lim\limits_{\delta t \to 0} \dfrac{\delta b}{\delta t}$

$= \dfrac{da}{dt} + \dfrac{db}{dt}$

Similarly, it can be proved that

$$\dfrac{d}{dt}(a - b) = \dfrac{da}{dt} - \dfrac{db}{dt}$$

Generalisation : $\dfrac{d}{dt}(a \pm b \pm c \pm ...) = \dfrac{da}{dt} \pm \dfrac{db}{dt} \pm \dfrac{dc}{dt} + ...$

(ii) Derivative of a scalar Multiple of a vector

$$\dfrac{d}{dt}(\phi a) = \phi \dfrac{da}{dt} + \dfrac{d\phi}{dt} a$$

Proof. $\dfrac{d}{dt}(\phi a) = \lim\limits_{\delta t \to 0} \dfrac{(\phi + \delta\phi)(a + \delta a) - \phi a}{\delta t}$

$= \lim\limits_{\delta t \to 0} \dfrac{\phi a + \phi\delta a + \delta\phi a + \delta\phi\delta a - \phi a}{\delta t}$

$= \lim\limits_{\delta t \to 0} \left[\phi \dfrac{\delta a}{\delta t} + \dfrac{\delta\phi}{\delta t} a + \dfrac{\delta\phi}{\delta t} \delta a \right]$

$= \lim\limits_{\delta t \to 0} \phi \dfrac{\delta a}{\delta t} + \lim\limits_{\delta t \to 0} \dfrac{\delta\phi}{\delta t} a + \lim\limits_{\delta t \to 0} \dfrac{\delta\phi}{\delta t} \delta a$

$= \phi \dfrac{da}{dt} + \dfrac{d\phi}{dt} a + \dfrac{d\phi}{dt} 0 \qquad\qquad [\because \ \delta t \to 0 \Rightarrow \delta a \Rightarrow 0]$

$= \phi \dfrac{da}{dt} + \dfrac{d\phi}{dt} a + 0 = \phi \dfrac{da}{dt} + \dfrac{d\phi}{dt} a$

(iii) Derivative of the Dot Product of two vectors

$$\dfrac{d}{dt}(a \cdot b) = a \cdot \dfrac{db}{dt} + \dfrac{da}{dt} \cdot b$$

Proof. $\dfrac{d}{dt}(a \cdot b) = \lim\limits_{\delta t \to 0} \dfrac{(a + \delta a)\cdot(b + \delta b) - a \cdot b}{\delta t}$

$= \lim\limits_{\delta t \to 0} \dfrac{a \cdot b + a \cdot \delta b + \delta a \cdot b + \delta a \cdot \delta b - a \cdot b}{\delta t}$

$= \lim\limits_{\delta t \to 0} \left[a \cdot \dfrac{\delta b}{\delta t} + \dfrac{\delta a}{\delta t} \cdot b + \dfrac{\delta a}{\delta t} \cdot \delta b \right]$

$$= \lim_{\delta t \to 0} a \cdot \frac{\delta b}{\delta t} + \lim_{\delta t \to 0} \frac{\delta a}{\delta t} \cdot b + \lim_{\delta t \to 0} \frac{\delta a}{\delta t} \cdot \delta b$$

$$= a \cdot \frac{db}{dt} + \frac{da}{dt} \cdot b + \frac{da}{dt} \cdot 0 \qquad\qquad [\because \ \delta t \to 0 \Rightarrow \delta b \to 0]$$

$$= a \cdot \frac{db}{dt} + \frac{da}{dt} \cdot b + 0 = a \cdot \frac{db}{dt} + \frac{da}{dt} \cdot b$$

(iv) Derivative of Cross Product of two vectors

$$\frac{d}{dt}(a \times b) = a \times \frac{db}{dt} + \frac{da}{dt} \times b$$

Proof. $\dfrac{d}{dt}(a \times b) = \lim\limits_{\delta t \to 0} \dfrac{(a + \delta a) \times (b + \delta b) - a \times b}{\delta t}$

$$= \lim_{\delta t \to 0} \frac{a \times b + a \times \delta b + \delta a \times b + \delta a \times \delta b - a \times b}{\delta t}$$

$$= \lim_{\delta t \to 0} \left[a \times \frac{\delta b}{\delta t} + \frac{\delta a}{\delta t} \times b + \frac{\delta a}{\delta t} \times \delta b \right]$$

$$= \lim_{\delta t \to 0} a \times \frac{\delta b}{\partial t} + \lim_{\delta t \to 0} \frac{\delta a}{\delta t} \times b + \lim_{\delta t \to 0} \frac{\delta a}{\delta t} \times \delta b$$

$$= a \times \frac{db}{dt} + \frac{da}{dt} \times b + \frac{da}{dt} \times 0$$

$$= a \times \frac{db}{dt} + \frac{da}{dt} \times b + 0 \qquad\qquad [\because \ \delta t \to 0 \Rightarrow \delta b \to 0$$

$$= a \times \frac{db}{dt} + \frac{da}{dt} \times b$$

(v) Derivative of Scalar Triple Product

$$\frac{d}{dt}[a\ b\ c] = \left[\frac{da}{dt}\ b\ c \right] + \left[a\ \frac{db}{dt}\ c \right] + \left[a\ b\ \frac{dc}{dt} \right]$$

Proof. $\dfrac{d}{dt}[a\ b\ c] = \dfrac{d}{dt}\{a \cdot (b \times c)\}$ \qquad [by the notation of *Scalar Triple Product*]

$$= a \cdot \frac{d}{dt}(b \times c) + \frac{da}{dt} \cdot (b \times c)$$

$$= a \cdot \left(b \times \frac{dc}{dt} + \frac{db}{dt} \times c \right) + \frac{da}{dt} \cdot (b \times c)$$

$$= a \cdot \left(b \times \frac{dc}{dt} \right) + a \cdot \left(\frac{db}{dt} \times c \right) + \frac{da}{dt} \cdot (b \times c)$$

$$= \left[a\, b\, \frac{dc}{dt} \right] + \left[a\, \frac{db}{dt}\, c \right] + \left[\frac{da}{dt}\, b\, c \right] \qquad \text{[by notation]}$$

$$= \left[\frac{da}{dt}\, b\, c \right] + \left[a\, \frac{db}{dt}\, c \right] + \left[a\, b\, \frac{dc}{dt} \right] \qquad \text{[in the cyclic order]}$$

(vi) Derivative of Vector Triple Product

$$\frac{d}{dt}\{a \times (b \times c)\} = \frac{da}{dt} \times (b \times c) + a \times \left(\frac{db}{dt} \times c \right) + a \times \left(b \times \frac{dc}{dt} \right)$$

Proof. $\dfrac{d}{dt}\{a \times (b \times c)\} = a \times \dfrac{d}{dt}(b \times c) + \dfrac{da}{dt} \times (b \times c)$

$$= a \times \left(b \times \frac{dc}{dt} + \frac{db}{dt} \times c \right) + \frac{da}{dt} \times (b \times c)$$

$$= a \times \left(b \times \frac{dc}{dt} \right) + a \times \left(\frac{db}{dt} \times c \right) + \frac{da}{dt} \times (b \times c)$$

$$= \frac{da}{dt} \times (b \times c) + a \times \left(\frac{db}{dt} \times c \right) + a \times \left(b \times \frac{dc}{dt} \right)$$

Derivative of the Function of a Function

Let r be a derivable function of scalar variable s and s be a derivable scalar function of scalar variable t.

Corresponding to a small increase δt in t, let δs and δr be the increase in s and r respectively, then

$$\delta t \to 0 \Rightarrow \delta s \to 0 \quad \text{and} \quad \delta r \to 0$$

Now $\dfrac{dr}{dt} = \lim\limits_{\delta t \to 0} \dfrac{\delta r}{\delta t} = \lim\limits_{\delta t \to 0} \left(\dfrac{\delta r}{\delta s} \dfrac{\delta s}{\delta t} \right)$

$$= \left(\lim_{\delta t \to 0} \frac{\delta r}{\delta s} \right) \left(\lim_{\delta t \to 0} \frac{\delta s}{\delta t} \right)$$

$$= \left(\lim_{\delta s \to 0} \frac{\delta r}{\delta s} \right) \left(\lim_{\delta t \to 0} \frac{\delta s}{\delta t} \right) \qquad [\delta t \to 0 \Rightarrow \delta s \to 0]$$

Hence $\dfrac{dr}{dt} = \dfrac{dr}{ds} \dfrac{ds}{dt}$

Some Important Theorems

Theorem : If a is a differentiable vector function of a scalar variable t and $|a| = a$; then :

(a) $\quad \dfrac{d}{dt}\left(a^2\right) = 2a\,\dfrac{da}{dt}$ 　　　　　　　(b) $\quad a \cdot \dfrac{da}{dt} = a\,\dfrac{da}{dt}$

Proof. (a) $\because \quad a^2 = a \cdot a = |a||a| \cos 0 = a^2$

$$\therefore \qquad \frac{d}{dt}\left(a^2\right) = \frac{d}{dt}\left(a^2\right) = 2a\,\frac{da}{dt}$$

(b) $\dfrac{d}{dt}(a^2) = \dfrac{d}{dt}(a \cdot a) = \dfrac{da}{dt} \cdot a + a \cdot \dfrac{da}{dt}$

$$= 2a \cdot \dfrac{da}{dt}$$

But by (a), $\dfrac{d}{dt}(a^2) = 2a\dfrac{da}{dt}$

$\therefore \qquad 2a \cdot \dfrac{da}{dt} = 2a\dfrac{da}{dt} \Rightarrow a \cdot \dfrac{da}{dt} = a\dfrac{da}{dt}$

Theorem : The necessary and sufficient condition for any vector a(t) to be a constant vector is that $\dfrac{da}{dt}$ = 0.

Proof. The condition is necessary ($\Rightarrow$) :

Let a(t) be a constant vector function of scalar variable t, then it will be proved that

$$\dfrac{da}{dt} = 0$$

$\because \qquad$ a(t) is a constant function, therefore a(t + δt) = a(t)

Hence $\dfrac{da}{dt} = \lim\limits_{\delta t \to 0} \dfrac{a(t + \delta t) - a(t)}{\delta t} = \lim\limits_{\delta t \to 0} \dfrac{0}{\delta t} = 0$

Therefore the condition is necessary i.e. the derivative of a constant vector is zero vector.

The condition of sufficient ($\Leftarrow$) :

Let $\dfrac{da}{dt}$ = 0, then it will be proved that a is constant vector.

Let $\qquad$ a(t) = a_1(t)i + a_2(t)j + a_3(t)k

then $\qquad \dfrac{da}{dt} = \dfrac{da_1}{dt}i + \dfrac{da_2}{dt}j + \dfrac{da_3}{dt}k$

Therefore

$$\dfrac{da}{dt} = 0 \Rightarrow \dfrac{da_1}{dt}i + \dfrac{da_2}{dt}j + \dfrac{da_3}{dt}k = 0$$

$\Rightarrow \qquad \dfrac{da_1}{dt} = 0, \dfrac{da_2}{dt} = 0, \dfrac{da_3}{dt} = 0$

$\Rightarrow \qquad a_1, a_2, a_3$ are constant vectors independent of t.

$\Rightarrow \qquad$ a(t) is a constant vector.

Therefore, the condition is sufficient i.e. $\dfrac{da}{dt}$ = 0

$\Rightarrow \qquad$ a is a constant vector.

Hence a(t) is a constant vector $\Leftrightarrow \dfrac{da}{dt}$ = 0

Theorem : The derivative of a vector of constant magnitude is perpendicular to that vector provided the vector itself is not a constant vector.

Proof. Let a be a vector of constant magnitude but it is not a constant vector i.e., |a| = a (constant) and $\dfrac{da}{dt} \neq 0$

then, $a \cdot a = a^2$ (constant) (as $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta) \Rightarrow \dfrac{d}{dt}(a \cdot a) = \dfrac{d}{dt}(a^2) = 0$

$\Rightarrow \qquad \dfrac{da}{dt} \cdot a + a \cdot \dfrac{da}{dt} = 0 \qquad \Rightarrow \qquad 2a \cdot \dfrac{da}{dt} = 0$

$\Rightarrow \qquad a \cdot \dfrac{da}{dt} = 0 \qquad \Rightarrow \qquad a \perp \dfrac{da}{dt} \qquad \left[\because \dfrac{da}{dt} \neq 0\right]$

Since $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta) \Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(90) \Rightarrow \vec{a} \cdot \vec{b} = 0$

Therefore the derivative of a is perpendicular to a.

Theorem : The necessary and sufficient condition that a(t) is a vector of constant magnitude is

$$a \cdot \dfrac{da}{dt} = 0$$

Proof. The condition is necessary ($\Rightarrow$) :

Let the magnitude of the vector a(t) is constant i.e., |a| = a (constant)

Then $a \cdot a = a^2$

$\Rightarrow \qquad \dfrac{d}{dt}(a \cdot a) = \dfrac{d}{dt}(a^2) = 0$

$\Rightarrow \qquad \dfrac{da}{dt} \cdot a + a \cdot \dfrac{da}{dt} = 0$

$\Rightarrow \qquad 2a \cdot \dfrac{da}{dt} = 0 \qquad \Rightarrow a \cdot \dfrac{da}{dt} = 0$

Therefore the given conditional is necessary.

The condition is sufficient ($\Leftarrow$) :

Let a(t) be a vector such that $a \cdot \dfrac{da}{dt} = 0$

Then it will be proved that the magnitude of a will be constant.

Now $a \cdot \dfrac{da}{dt} = 0$

$\Rightarrow \qquad a \cdot \dfrac{da}{dt} + \dfrac{da}{dt} \cdot a = 0$

$\Rightarrow \qquad \dfrac{d}{dt}(a \cdot a) = 0 \Rightarrow a \cdot a$ is constant.

$\Rightarrow \qquad a^2$ is constant $\Rightarrow$ |a| is constant.

Therefore, the given condition is sufficient.

Hence $|a(t)|$ is constant $\Leftrightarrow a \cdot \dfrac{da}{dt} = 0$

Theorem : The necessary and sufficient condition for the vector $a(t)$ to have constant direction is

$$a \times \frac{da}{dt} = 0$$

Proof. First we prove a *lemma*

Let $\hat{a}$ be unit vector along a and its magnitude be a, then by definition,

$$a = a\hat{a}$$

$$\Rightarrow \qquad \frac{d}{dt}(a) = \frac{d}{dt}(a\hat{a}) = a\frac{d\hat{a}}{dt} + \frac{da}{dt}\hat{a} \qquad \qquad ...(1)$$

Multiplying both sides of (1) vectorially by vector a,

$$a \times \frac{da}{dt} = (a\ \hat{a}) \times \left(a\frac{d\hat{a}}{dt} + \frac{da}{dt}\hat{a} \right)$$

$$= a^2\hat{a} \times \frac{d\hat{a}}{dt} + a\frac{da}{dt}(\hat{a} \times \hat{a})$$

$$= a^2\hat{a} \times \frac{d\hat{a}}{dt} \qquad \qquad \left[\because \hat{a} \times \hat{a} = 0 \right] \qquad ...(2)$$

The condition is necessary ($\Rightarrow$) :

Let the direction of the vector $a(t)$ be constant. Then $\hat{a}$ will be a constant vector because its magnitude is 1 (constant).

Hence $\dfrac{d\hat{a}}{dt} = 0$

Therefore in this case, from (2)

$$a \times \frac{da}{dt} = a^2\hat{a} \times \frac{d\hat{a}}{dt} = a^2\hat{a} \times 0 = 0$$

which shows that the given condition is necessary.

The condition is sufficient ($\Leftarrow$) :

Let $a(t)$ be a vector such that $a \times \dfrac{da}{dt} = 0$ $\qquad \qquad ...(3)$

then we shall prove that the direction of a is constant

Now $\quad a \times \dfrac{da}{dt} = 0 \quad \Rightarrow a^2\hat{a} \times \dfrac{d\hat{a}}{dt} = 0$ $\qquad \qquad$ [by (2)]

$$\Rightarrow \hat{a} \times \frac{d\hat{a}}{dt} = 0 \Rightarrow \hat{a} \,\|\, \frac{d\hat{a}}{dt} \qquad \qquad ...(4)$$

Again the magnitude of $\hat{a}$ is 1, (constant).

Therefore by theorem,

$$\hat{a} \cdot \frac{d\hat{a}}{dt} = 0 \Rightarrow \hat{a} \perp \frac{da}{dt} \qquad \qquad ...(5)$$

(4) and (5) show that the vectors $\hat{a}$ and $\dfrac{d\hat{a}}{dt}$ are mutually parallel as well as perpendicular (contradiction), which is possible only when $\dfrac{d\hat{a}}{dt} = 0$

Since $\hat{a} \neq 0$, $\hat{a}$ will be a constant vector, being $\dfrac{d\hat{a}}{dt} = 0$,

i.e., its direction will also be constant.

Consequently the direction of a(t) will also be constant.

Therefore the condition is sufficient.

Hence the direction of a(t) is constant $\Leftrightarrow$ a $\times$ $\dfrac{da}{dt} = 0$

Unit tangent vector to a curve at a point

If r = f(t) is a single valued continuous and differentiable function of a scalar variable t, then we have seen that $\dfrac{dr}{dt}$ is a vector whose direction is along the tangent at the point 't' to the curve r = f(t).

Therefore, at the point 't'

Tangential unit vector $\hat{n} = \dfrac{dr/dt}{|dr/dt|}$

Instantaneous Velocity and Acceleration :

If t is the time and $r = \overrightarrow{OP}$ is the position vector of any moving point P wrt the origin O for the vector r = f(t), then

δr = displacement of the particle in the interval δt

$\therefore$ $\dfrac{\delta r}{\delta t}$ = average velocity of the particle in the interval δt

Hence if the **velocity vector** of any particle at the point P is v,

then $\quad v = \lim\limits_{\delta t \to 0} \dfrac{\delta r}{\delta t}$ or $v = \dfrac{dr}{dt}$ ...(1)

Since the direction of the vector $\dfrac{dr}{dt}$ is along the tangent at P on the curve, therefore the direction of the velocity of the particle at P will also be along the tangent at P on the curve.

Again if δv be the change in the velocity v of the particle in the time interval δt, then

$\dfrac{\delta v}{\delta t}$ = average acceleration of the particle in the interval δt

Hence if the **acceleration vector** of any particle at the point P is f,

then $\quad f = \lim\limits_{\delta t \to 0} \dfrac{\delta v}{\delta t} = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{dr}{dt}\right)$ or $f = \dfrac{d^2r}{dt^2}$ ...(2)

Example : If r = a cos t i + a sin t j + t k, find the following :

(a) $\dfrac{dr}{dt}$ (b) $\left|\dfrac{dr}{dt}\right|$

(c) $\dfrac{d^2r}{dt^2}$ (d) $\left|\dfrac{d^2r}{dt^2}\right|$

Solution. (a) Given r = a cos t i + a sin t j + t k

$\therefore$ $\dfrac{dr}{dt} = \dfrac{d}{dt}(a\cos t)i + \dfrac{d}{dt}(a\sin t)j + \dfrac{d}{dt}(t)k$ = – a sin t i + a cos t j + k

(b) $\left|\dfrac{dr}{dt}\right| = \sqrt{\left\{(-a\sin t)^2 + (a\cos t)^2 + (1)^2\right\}}$

           $= \sqrt{(a^2\sin^2 t + a^2\cos^2 t + 1)} = \sqrt{(a^2 + 1)}$

(c) $\dfrac{d^2r}{dt^2} = \dfrac{d}{dt}\left[\dfrac{dr}{dt}\right] = \dfrac{d}{dt}\left[-a\sin t\,i + a\cos t\,j + k\right]$

           $= \dfrac{d}{dt}(-a\sin t)t + \dfrac{d}{dt}(a\cos t)j + \dfrac{d}{dt}(1)k$

           = –a cos t i – a sin t j + 0k

           = –a(cos t i + sin t j)

(d) $\left|\dfrac{d^2r}{dt^2}\right| = \sqrt{\left\{(-a\cos t)^2 + (-a\sin t)^2\right\}}$

           $= \sqrt{(a^2\cos^2 t + a^2\sin^2 t)} = a$

Partial Derivatives of Vectors

Let r = f(x, y, z,....) be a vector function of independent variables x, y, z, ... Let δr be the change in r due to the small change δx in x, whereas there is no change in other independent variables, then

if $\lim\limits_{\delta x \to 0}\dfrac{\delta r}{\delta x}$ exist, is called the **partial derivative** of the vector r wrt x and is denoted by $\dfrac{\partial r}{\partial x}$.

Therefore, $\dfrac{\partial r}{\partial x} = \lim\limits_{\delta x \to 0}\dfrac{\delta r}{\delta x} = \lim\limits_{\delta x \to 0}\dfrac{f(x + \delta x, y, z, ...) - f(x, y, z, ...)}{\delta x}$

Clearly $\dfrac{\partial r}{\partial x}$ is the *ordinary* differential coefficient or *derivative* of r wrt x and all other independent variables are treated as constants.

Similarly, the partial derivatives of r wrt other independent variables y, z, ... etc. can also be defined and denoted respectively by

$\dfrac{\partial r}{\partial y}, \dfrac{\partial r}{\partial z}$ ... etc.

Successive partial derivatives of a vector function r can also be defined as in calculus of scalar variables :

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right), \frac{\partial^2 r}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial r}{\partial y}\right), \frac{\partial^2 r}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial r}{\partial z}\right)$$

$$\frac{d^2 r}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial y}\right), \frac{\partial^2 r}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial r}{\partial x}\right) \text{etc.}$$

If second order partial derivative of r are *continuous*, then

$$\frac{\partial^2 r}{\partial x \partial y} = \frac{\partial^2 r}{\partial y \partial y} \quad \text{i.e., } \textbf{commutative.}$$

If a and b are two vector functions of independent variables x, y, z and ϕ is a scalar function of x, y, z, then the following results can be easily proved:

(1) $\quad \dfrac{\partial}{\partial x}(a \pm b) = \dfrac{\partial a}{\partial x} \pm \dfrac{\partial b}{\partial x}$

(2) $\quad \dfrac{\partial}{\partial x}(\phi.a) = \phi\dfrac{\partial a}{\partial x} + \dfrac{\partial \phi}{\partial x}a$

(3) $\quad \dfrac{\partial}{\partial x}(a \cdot b) = a \cdot \dfrac{\partial b}{\partial x} + \dfrac{\partial a}{\partial x} \cdot b$

(4) $\quad \dfrac{\partial}{\partial x}(a \times b) = a \times \dfrac{\partial b}{\partial x} + \dfrac{\partial a}{\partial x} \times b$

(5) $\quad \dfrac{\partial}{\partial x}(a_1 i + a_2 j + a_3 k) = \dfrac{\partial a_1}{\partial x}i + \dfrac{\partial a_2}{\partial x}j + \dfrac{\partial a_3}{\partial x}k,$

where a_1, a_2, a_3 are scalar functions of independent variables x, y, z.

Example : If r = $(2x^2y - x^4)i - (e^{xy} - y \sin x)j + (x^2 \cos y)k$, find the following :

(a) $\quad \dfrac{\partial r}{\partial x}$ $\qquad$ (b) $\quad \dfrac{\partial r}{\partial y}$ $\qquad$ (c) $\quad \dfrac{\partial^2 r}{\partial x^2}$

(c) $\quad \dfrac{\partial^2 r}{\partial y^2}$ $\qquad$ (e) $\quad \dfrac{\partial^2 r}{\partial x \partial y}$ $\qquad$ (f) $\quad \dfrac{\partial^2 r}{\partial y \partial x}$

Also show that : $\dfrac{\partial^2 r}{\partial x \partial y} = \dfrac{\partial^2 r}{\partial y \partial x}$

Solution. $\because$ r = $(2x^2y - x^4)i - (e^{xy} - y \sin x)j + (x^2 \cos y)$ k

$\therefore \quad \dfrac{\partial r}{\partial x} = \dfrac{\partial}{\partial x}(2x^2y - x^4)i - \dfrac{\partial}{\partial x}(e^{xy} - y\sin x)j + \dfrac{\partial}{\partial x}(x^2\cos y)k$

$= (4xy - 4x^3)i - (ye^{xy} - y\cos x)j + (2x\cos y)k$ $\qquad$ ...(1)

Similarly,

$\dfrac{\partial r}{\partial y} = \dfrac{\partial}{\partial y}(2x^2y - x^4)i - \dfrac{\partial}{\partial y}(e^{xy} - y\sin x)j + \dfrac{\partial}{\partial y}(x^2\cos y)k$

$$= \left(2x^2\right)i - \left(xe^{xy} - \sin x\right)j - \left(x^2 \sin y\right)k \qquad ...(2)$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left[\left(4xy - 4x^3\right)i - \left(ye^{xy} - y\cos x\right)j + \left(2x\cos y\right)k\right] \qquad \text{[by (1)]}$$

$$= \left(4y - 12x^2\right)i - \left(y^2 e^{xy} + y\sin x\right)j + \left(2\cos y\right)k \qquad ...(3)$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial r}{\partial y}\right) = \frac{\partial}{\partial y}\left[2x^2 i - \left(xe^{xy} - \sin x\right)j - x^2 \sin y k\right] \qquad \text{[by (2)]}$$

$$= -x^2 e^{xy} j - x^2 \cos y k \qquad ...(4)$$

$$\frac{\partial^2 r}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial r}{\partial y}\right] = \frac{\partial}{\partial x}\left[2x^2 i - \left(xe^{xy} - \sin x\right)j - x^2 \sin y k\right] \qquad \text{[by (2)]}$$

$$= 4xi - (e^{xy} + xye^{xy} - \cos x)j - 2x \sin y k \qquad ...(5)$$

$$\frac{\partial^2 r}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial r}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left[\left(4xy - 4x^3\right)i - \left(ye^{xy} - y\cos x\right)j + 2x\cos y k\right] \qquad \text{[by(1)]}$$

$$= 4xi - (e^{xy} + xye^{xy} - \cos x)j - 2x \sin y k \qquad ...(6)$$

From (5) and (6), the required result is verified.

Example : If a = x²yz i – 2xz³ j + xz² k; b = 2zi + yj – x²k ;

Find : $\left[\frac{\partial^2}{\partial x \partial y}(a \times b)\right]_{(1,0,-2)}$

Solution. $\frac{\partial^2}{\partial x \partial y}(a \times b) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}(a \times b)\right] = \frac{\partial}{\partial x}\left[a \times \frac{\partial b}{\partial y} + \frac{\partial a}{\partial y} \times b\right]$

$$= \frac{\partial}{\partial x}\left(a \times \frac{\partial b}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{\partial a}{\partial y} \times b\right)$$

$$= a \times \frac{\partial^2 b}{\partial x \partial y} + \frac{\partial a}{\partial x} \times \frac{\partial b}{\partial y} + \frac{\partial a}{\partial y} \times \frac{\partial b}{\partial x} + \frac{\partial^2 a}{\partial x \partial y} \times b \qquad ...(1)$$

or first calculate a × b by

$$\begin{vmatrix} i & J & K \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

if $\quad a = a_1 i + a_2 J + a_3 k$

$\quad\quad b = b_1 i + b_2 J + b_3 k$

then first differentiate (a × b) by x and then by y.

Now $\dfrac{\partial \mathbf{a}}{\partial x} = 2xyz\,\mathbf{i} - 2z^3\mathbf{j} + z^2\mathbf{k};$ $\qquad \dfrac{\partial \mathbf{b}}{\partial x} = -2x\,\mathbf{k}$

$\dfrac{\partial \mathbf{a}}{\partial y} = x^2z\,\mathbf{i};$ $\qquad\qquad\qquad\qquad \dfrac{\partial \mathbf{b}}{\partial y} = \mathbf{j}$

$\dfrac{\partial^2 \mathbf{a}}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial \mathbf{a}}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(x^2z\,\mathbf{i}\right) = 2xz\,\mathbf{i}$

$\dfrac{\partial^2 \mathbf{b}}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial \mathbf{b}}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(\mathbf{j}\right) = 0$

$\therefore \quad \mathbf{a} \times \dfrac{\partial^2 \mathbf{b}}{\partial x \partial y} = \mathbf{a} \times 0 = 0$ $\hfill$ ...(2)

$\dfrac{\partial \mathbf{a}}{\partial x} \times \dfrac{\partial \mathbf{b}}{\partial y} = \left(2xyz\,\mathbf{i} - 2z^3\mathbf{j} + z^2\mathbf{k}\right) \times \mathbf{j} = 2xyz\,\mathbf{k} - z^2\mathbf{i}$ $\hfill$ ...(3)

$\vec{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$\vec{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

Since $\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}(a_2 b_3 - a_3 b_2) - \mathbf{j}(a_1 b_3 - a_3 b_1) + \mathbf{k}(a_1 b_2 - a_2 b_1)$

$\dfrac{\partial \mathbf{a}}{\partial y} \times \dfrac{\partial \mathbf{b}}{\partial x} = \left(x^2z\,\mathbf{i}\right) \times \left(-2x\,\mathbf{k}\right) = 2x^3z\,\mathbf{j}$ $\hfill$ ...(4)

and $\dfrac{\partial^2 \mathbf{a}}{\partial x \partial y} \times \mathbf{b} = 2xz\,\mathbf{i} \times \left(2z\,\mathbf{i} + y\,\mathbf{j} - x^2\mathbf{k}\right) = 2xyz\,\mathbf{k} + 2x^3z\,\mathbf{j}$

Substituting the values from (2), (3), (4), (5) in (1),

$\dfrac{\partial^2}{\partial x \partial y}(\mathbf{a} \times \mathbf{b}) = 0 + \left(2xyz\,\mathbf{k} - z^2\mathbf{i}\right) + \left(2x^3z\,\mathbf{j}\right) + \left(2xyz\,\mathbf{k} + 2x^3z\,\mathbf{j}\right)$

$= -z^2\mathbf{i} + 4x^3z\,\mathbf{j} + 4xyz\,\mathbf{k}$

$\therefore \quad$ At the point (1, 0, –2),

$\dfrac{\partial^2}{\partial x \partial y}(\mathbf{a} \times \mathbf{b}) = -4\mathbf{i} - 8\mathbf{j} + 0\mathbf{k} = -4\mathbf{i} - 8\mathbf{j}$

The Vector Differential Operator : [Del (∇)]

The vector operator ∇ is defined as follows :

$$\nabla \equiv \mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z} \equiv \dfrac{\partial}{\partial x}\mathbf{i} + \dfrac{\partial}{\partial y}\mathbf{j} + \dfrac{\partial}{\partial z}\mathbf{k}$$

This is a differential operator which is read as *Del* or ' *Nabla* '.

Scalar Point Function :

Let R be a region of points in the space. If a scalar quantity $\phi(P)$ or $\phi(x, y, z)$ is obtained corresponding to every point $P(x, y, z)$ by some rule, then ϕ is called a *scalar point function* in the region R.

Example : The quantity of any thing.

Example : Density

Example : Temperature (at any instant).

Scalar Field : The set of points in any region R and the vector function φ at those points together is called a *scalar field.*

Example : Temperature distribution in any medium.

Example : Gravitational potential of a system of masses.

Example : Electrostatic potential of charges of a system.

Vector Point Function :

Let R be a region of points in the space. If a scalar quantity V(P) or V(x, y, z) is obtained corresponding to every point P(x, y, z) by some rule, then V is called a *vector point function* in the region R.

Example : Velocity of any moving point of fluid at any instant.

Example : Force of electrical or magnetic intensity of any point of the electrical or magnetic field.

Vector Field : A vector field is a vector each of whose components is a scalar field, that is, a function of our variables. We use any of the following notations for one :

$$v(x, y, z) = (v_1(x, y, z), v_2(x, y, z), v_3(x, y, z))$$
$$v(x, y, z) = v_1(x, y, z)i + v_2(x, y, z)j + v_3(x, y, z)k$$

Example : Velocity of moving fluid at any instant

Example : Force of electricity intensity

Example : Force of magnetic intensity.

3. GRADIENT OF A SCALAR POINT FUNCTION

Definition : If f(x, y, z) is a continuous differentiable scalar point function at every point (x, y, z), then gradient of f is expressed as grad f and defined as follows :

$$\text{grad } f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right)f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} \qquad \text{...(1)}$$

The operation ∇ : *Definition* :

$$\nabla \equiv \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k \equiv i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \qquad \text{...(2)}$$

This operator is called *Del* or *Nabla.*

Therefore $\qquad \nabla f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right)f \qquad \text{...(3)}$

$\therefore \qquad$ grad f = ∇f

i.e. $\qquad \nabla f = \sum i\frac{\partial f}{\partial x}$

By the definition, it is clear that the gradient of any scalar point function f i.e. grad f is a vector whose components parallel to the co-ordinate axes are respectively $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$.

Theorems on Gradient

Theorem : If f and g are scalar point function, then :

(a) grad (f ± g) = grad f ± grad g

(b) grad (f g) = f(grad g) + g(grad f)

(c) $\quad \text{grad}\left(\frac{f}{g}\right) = \frac{g(\text{grad } f) - f(\text{grad } g)}{g^2}, g \neq 0$

Proof. (a) grad(f ± g) = ∇(f ± g) = $\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right)$(f ± g)

$$= i\frac{\partial}{\partial x}(f \pm g) + j\frac{\partial}{\partial y}(f \pm g) + k\frac{\partial}{\partial z}(f \pm g)$$

$$= i\frac{\partial f}{\partial x} \pm i\frac{\partial g}{\partial x} + j\frac{\partial f}{\partial y} \pm j\frac{\partial g}{\partial y} + k\frac{\partial f}{\partial z} \pm k\frac{\partial g}{\partial z}$$

$$= \left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} \right) \pm \left(i\frac{\partial g}{\partial x} + j\frac{\partial g}{\partial y} + k\frac{\partial g}{\partial z} \right)$$

$$= \left(i\frac{\partial}{\partial y} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right)f \pm \left(i\frac{\partial}{\partial y} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right)g$$

= grad f ± grad g

$\therefore \qquad \nabla(f \pm g) = \nabla f \pm \nabla g$

(b) $\text{grad}(fg) = \nabla(fg) = \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z}\right)(fg)$

$$= i\dfrac{\partial}{\partial x}(fg) + j\dfrac{\partial}{\partial y}(fg) + k\dfrac{\partial}{\partial z}(fg)$$

$$= i\left(f\dfrac{\partial g}{\partial x} + g\dfrac{\partial f}{\partial x}\right) + j\left(f\dfrac{\partial g}{\partial y} + g\dfrac{\partial f}{\partial y}\right) + k\left(f\dfrac{\partial g}{\partial z} + g\dfrac{\partial f}{\partial z}\right)$$

$$= f\left(i\dfrac{\partial g}{\partial x} + j\dfrac{\partial g}{\partial y} + k\dfrac{\partial g}{\partial z}\right) + g\left(i\dfrac{\partial f}{\partial x} + j\dfrac{\partial f}{\partial y} + k\dfrac{\partial f}{\partial z}\right)$$

$$= f(\text{grad } g) + g(\text{grad } f)$$

$\therefore$ $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(c) $\text{grad}\left(\dfrac{f}{g}\right) = \nabla\left(\dfrac{f}{g}\right) = \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z}\right)\left(\dfrac{f}{g}\right)$

$$= i\dfrac{\partial}{\partial x}\left(\dfrac{f}{g}\right) + j\dfrac{\partial}{\partial y}\left(\dfrac{f}{g}\right) + k\dfrac{\partial}{\partial z}\left(\dfrac{f}{g}\right)$$

$$= \Sigma i\dfrac{\partial}{\partial x}\left(\dfrac{f}{g}\right) = \Sigma i\left(\dfrac{g\dfrac{\partial f}{\partial x} - f\dfrac{\partial g}{\partial x}}{g^2}\right)$$

$$= \dfrac{1}{g^2}\Sigma i\left(g\dfrac{\partial f}{\partial x} - f\dfrac{\partial g}{\partial x}\right)$$

$$= \dfrac{1}{g^2}\left[i\left(\dfrac{\partial f}{\partial x} - f\dfrac{\partial f}{\partial x}\right) + j\left(g\dfrac{\partial f}{\partial y} - f\dfrac{\partial f}{\partial y}\right) + k\left(f\dfrac{\partial f}{\partial z} - f\dfrac{\partial f}{\partial z}\right)\right]$$

$$= \dfrac{1}{g^2}\left[g\left(i\dfrac{\partial f}{\partial x} + j\dfrac{\partial f}{\partial y} + k\dfrac{\partial f}{\partial z}\right) - f\left(i\dfrac{\partial g}{\partial x} + j\dfrac{\partial g}{\partial y} + k\dfrac{\partial g}{\partial z}\right)\right]$$

$$= \dfrac{1}{g^2}\left[g(\text{grad } f) - f(\text{grad } g)\right]$$

$\therefore$ $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g(\nabla f) - f(\nabla g)}{g^2}$

Gradient as Surface Normal Vector

The family of the surface given by

$$f(x, y, z) = c$$

with various values of c are called the level surfaces of the scalar function f. These surfaces are also called *equipotential* or iso surfaces e.g. isothermal surfaces.
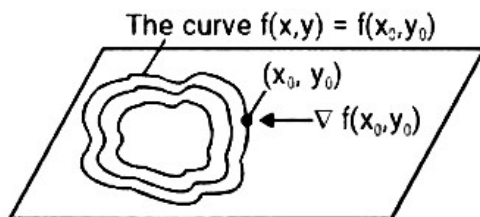
Fig. The gradient of a differentiable function of two variable at a point is always normal to the function's level curve through that point.

Let $r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be a smooth curve on the level surface $f(x, y, z) = c$ of a differentiable function f. Then

$$f(x(t), y(t), z(t)) = C$$

Differentiating both sides of w.r.t x,

$$\frac{d}{dt}f[x(t), y(t), z(t)] = \frac{dc}{df} = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$$

$$\Rightarrow \quad \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) \cdot \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) = 0$$

$$\Rightarrow \quad \nabla f \cdot \frac{dr}{dt} = 0$$

Hence ∇f is orthogonal to all vectors $\frac{dr}{dt} = \dot{r}$ in the tangent plane, i.e. ∇f is normal to the surface at every point along the curve.
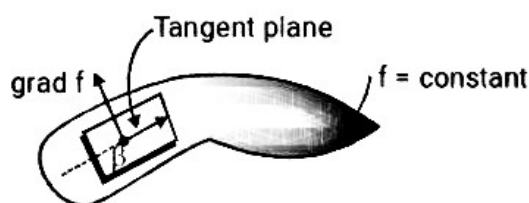
Fig.

The angle between any two surfaces of $f(x, y, z) = c_1$ and $g(x, y, z) = c_2$ is the angle between their corresponding normal given by ∇f and ∇g respectively.

Let θ be the angle between the two surfaces. Then

$$\cos\theta = \frac{\nabla f \cdot \nabla g}{|\nabla f| \cdot |\nabla g|}$$

The angle between two planes is obtained from the angle between their normals
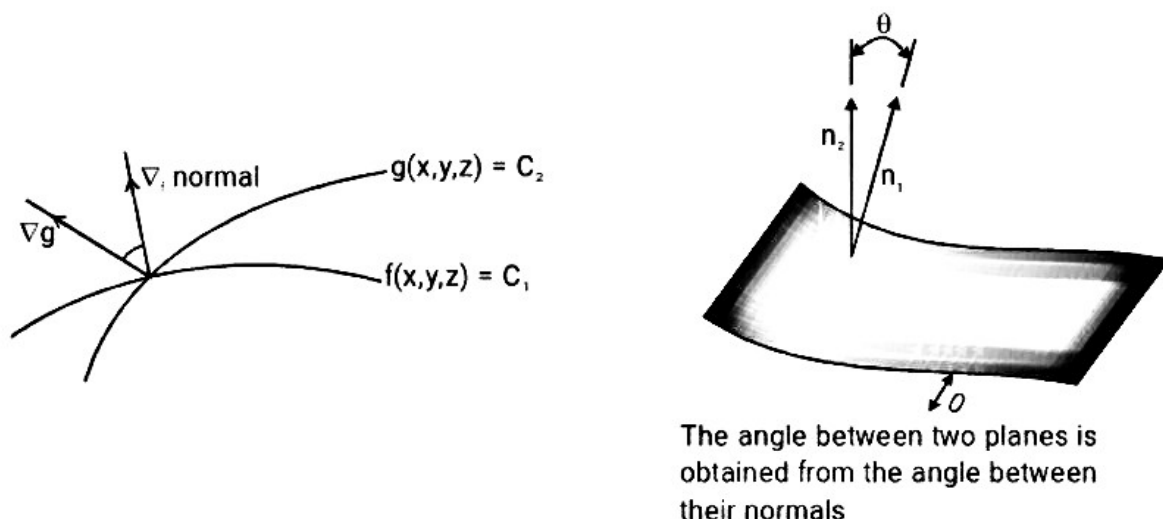
Fig.

Theorem : A scalar point function f is constant iff grad f = 0

Proof. Let f be a constant scalar point function, then

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0$$

$\therefore$ grad $f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = i(0) + j(0) + k(0) = 0$

Conversely : Let grad f = 0, then

grad $f = 0 \Rightarrow i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = 0$

$\Rightarrow \quad \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0$

$\Rightarrow \quad$ f is independent of x, y and z.

$\Rightarrow \quad$ f is constant scalar point function.

Hence f is a constant scalar point function $\Leftrightarrow \nabla f = 0$.

Operator a·∇

If $a = a_1 i + a_2 j + a_3 k$, then the operator a·∇ is defined as :

$$a\cdot\nabla = a\cdot i\frac{\partial}{\partial x} + a\cdot j\frac{\partial}{\partial y} + a\cdot k\frac{\partial}{\partial z}$$

$$= a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$$

i.e. $a\cdot\nabla = \Sigma(a\cdot i)\frac{\partial}{\partial x} = \Sigma a_1\frac{\partial}{\partial x}$

Therefore if f is any scalar point function, then

$$(a\cdot\nabla)f = \left(a\cdot i\frac{\partial}{\partial x} + a\cdot j\frac{\partial}{\partial y} + a\cdot k\frac{\partial}{\partial z}\right)f$$

$$= (a \cdot i)\frac{\partial f}{\partial x} + (a \cdot j)\frac{\partial f}{\partial y} + (a \cdot k)\frac{\partial f}{\partial z}$$

$$= a_1\frac{\partial f}{\partial x} + a_2\frac{\partial f}{\partial y} + a_3\frac{\partial f}{\partial z}$$ ...(1)

Again $a \cdot \nabla f = (a_1 i + a_2 j + a_3 k) \cdot \left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} \right)$

$$= a_1\frac{\partial f}{\partial x} + a_2\frac{\partial f}{\partial y} + a_3\frac{\partial f}{\partial z}$$ ...(2)

Thus from (1) and (2), we have $(a \cdot \nabla)f = a \cdot \nabla f$

Example : If $f(x, y, z) = 3x^2 y - y^3 z^2$;
find the value of grad f at the point $(1, -2, -1)$.

Solution. grad $f = \nabla f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right)\left(3x^2 y - y^3 z^2 \right)$

$$= i\frac{\partial}{\partial x}\left(3x^2 y - y^3 z^2 \right) + j\frac{\partial}{\partial y}\left(3x^2 y - y^3 z^2 \right) + k\frac{\partial}{\partial z}\left(3x^2 y - y^3 z^2 \right)$$

$$= i(6xy) + j\left(3x^2 - 3y^2 z^2 \right) + k\left(-2y^3 z \right)$$

$$= 6xy\,i + \left(3x^2 - 3y^2 z^2 \right)j - 2y^3 z\,k$$

Substituting $x = 1$, $y = -2$, $z = -1$,
grad $f = 6(1)(-2)i + (3 - 3.4.1)j - 2(-8)(-1)k$
$= -12i - 9j - 16k$

Example : If $r = |r|$, where $r = xi + yj + zk$, prove that :

(a) grad $r = \hat{r}$ (b) grad$\left(\dfrac{1}{r}\right) = -\dfrac{r}{r^3}$

(c) $\nabla f(r) = f'(r)\nabla r$ (d) $\nabla f(r) \times r = 0$

(e) grad $r^m = m r^{m-2} r$

Solution. $\because$ $r^2 = x^2 + y^2 + z^2$

$\therefore$ $2r\dfrac{\partial r}{\partial x} = 2x \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

(a) grad $r = \nabla r = i\dfrac{\partial r}{\partial x} + j\dfrac{\partial r}{\partial y} + k\dfrac{\partial r}{\partial z} = i\left(\dfrac{x}{r}\right) + j\left(\dfrac{y}{r}\right) + k\left(\dfrac{z}{r}\right)$

$$= \frac{1}{r}(xi + yj + zk) = \frac{1}{r}r = \hat{r}. \qquad \left[\hat{r} \text{ is unit vector i.e. } \hat{r} = \frac{\vec{r}}{|\vec{r}|} \right]$$

(b) $\quad \text{grad}\left(\dfrac{1}{r}\right) = \nabla\left(\dfrac{1}{r}\right) = i\dfrac{\partial}{\partial x}\left(\dfrac{1}{r}\right) + j\dfrac{\partial}{\partial y}\left(\dfrac{1}{r}\right) + k\dfrac{\partial}{\partial z}\left(\dfrac{1}{r}\right)$

$$= i\left(-\dfrac{1}{r^2}\dfrac{\partial r}{\partial x}\right) + j\left(-\dfrac{1}{r^2}\dfrac{\partial r}{\partial y}\right) + k\left(-\dfrac{1}{r^2}\dfrac{\partial r}{\partial z}\right)$$

$$= -\dfrac{1}{r^2}\left[i\left(\dfrac{x}{r}\right) + j\left(\dfrac{y}{r}\right) + k\left(\dfrac{z}{r}\right)\right]$$

$$= -\dfrac{1}{r^3}\left(xi + yj + zk\right) = -\dfrac{1}{r^3}r$$

(c) $\quad \nabla f(r) = \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z}\right)f(r)$

$$= i\dfrac{\partial}{\partial x}f(r) + j\dfrac{\partial}{\partial y}f(r) + k\dfrac{\partial}{\partial z}f(r)$$

$$= if'(r)\dfrac{\partial r}{\partial x} + jf'(r)\dfrac{\partial r}{\partial y} + kf'(r)\left(\dfrac{\partial r}{\partial z}\right)$$

$$= f'(r)\left[i\dfrac{\partial r}{\partial x} + j\dfrac{\partial r}{\partial y} + k\dfrac{\partial r}{\partial z}\right]$$

$$= f'(r)\nabla r$$

(d) Earlier it has been proved that

$$\nabla f(r) = f'(r)\nabla(r) \quad \text{and} \quad \nabla r = \dfrac{1}{r}r$$

$$\therefore \quad \nabla f(r) \times r = \left[f'(r)\dfrac{1}{r}r\right] \times r = \left[\dfrac{1}{r}f'(r)\right](r \times r)$$

$$= \left[\dfrac{1}{r}f'(r)\right]0 = 0$$

Since $\vec{a} \times \vec{a} = |\vec{a}|.|\vec{a}|\sin\theta\,\hat{n}$ and the angle between same vector is zero so

$\sin\theta = \sin(0) = 0$

$\Rightarrow \quad \boxed{\vec{a} \times \vec{a} = 0}$

(e) $\quad \text{grad } r^m = \nabla r^m = i\dfrac{\partial r^m}{\partial x} + j\dfrac{\partial r^m}{\partial y} + k\dfrac{\partial r^m}{\partial z}$

$$= i\left(mr^{m-1}\dfrac{\partial r}{\partial x}\right) + j\left(mr^{m-1}\dfrac{\partial r}{\partial y}\right) + k\left(mr^{m-1}\dfrac{\partial r}{\partial z}\right)$$

$$= mr^{m-1}\left[i\left(\frac{x}{r}\right)+j\left(\frac{y}{r}\right)+k\left(\frac{z}{r}\right)\right]$$

$$= mr^{m-2}(xi + yj + zk) = mr^{m-2}\ r$$

Example : If a, b be any constant vectors and r = x i + y j + z k; then prove that :

(a) $\nabla(a \cdot r) = a$ (b) $\nabla[r\ a\ b] = a \times b$

Solution. (a) Let a = $a_1i + a_2j + a_3k$, where a_1, a_2, a_3 are constants.

Then   a·r = $(a_1i + a_2j + a_3k) \cdot (xi + yj + zk)$

$$= a_1x + a_2y + a_3z$$

$$\therefore \quad \nabla(a\cdot r) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$$

$$= \Sigma i\frac{\partial}{\partial x}(a_1x + a_2y + a_3z) = \Sigma i(a_1)$$

$$= i(a_1) + j(a_2) + k(a_3) = a_1i + a_2j + a_3k$$

$$= a.$$

(b) $\nabla[r\ a\ b] = \nabla[r \cdot (a \times b)]$ [by notation]

$\quad\quad = \nabla[r \cdot A]$, where A = a × b is a constant vector.

$\quad\quad = \nabla[A \cdot r] = A$ [by (a)]

$\quad\quad = a \times b.$

Example : If u = x + y + z, v = $x^2 + y^2 + z^2$ and w = yz + zx + xy prove that :

(grad u) · {grad v × grad w} = 0

Solution. From u = x + y + z, $\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 1$ and $\frac{\partial u}{\partial z} = 1$

$$\therefore \quad \text{grad u} = i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}$$

$$= i(1) + j(1) + k(1) = i + j + k \quad\quad\quad\quad\quad ...(1)$$

Again from v = $x^2 + y^2 + z^2$, $\frac{\partial v}{\partial x} = 2x, \frac{\partial v}{\partial y} = 2y$ and $\frac{\partial v}{\partial z} = 2z$

$$\therefore \quad \text{grad v} = i\frac{\partial v}{\partial x} + j\frac{\partial v}{\partial y} + k\frac{\partial v}{\partial z}$$

$$= i(2x) + j(2y) + k(2z) = 2(xi + yj + zk) \quad\quad ...(2)$$

and from   w = yz + zx + xy,

$$\frac{\partial w}{\partial x} = y + z, \frac{\partial w}{\partial y} = z + x \text{ and } \frac{\partial w}{\partial z} = x + y$$

$$\therefore \quad \text{grad w} = i\frac{\partial w}{\partial x} + j\frac{\partial w}{\partial y} + k\frac{\partial w}{\partial z}$$

$$= i(y + z) + j(z + x) + k(x + y)$$

$$= (y + z)i + (z + x)j + (x + y)k \quad\quad\quad\quad ...(3)$$

Now from (1), (2) and (3),

$$(\text{grad } u) \cdot \{(\text{grad } v) \times (\text{grad } w)\} = \text{grad } u. \begin{vmatrix} i & j & k \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2\,[y(x+y) - z(z+x)]\,i - 2\,[x(x+y) + z(y+z)]\,j + 2\,[x(x+z) - y(y+z)]k$$

Simplify the above equation, we will get

$$= 0$$

Directional Derivative : Definition :

Let $f(x, y, z)$ be a scalar point function in any region R and at any pint $P(x, y, z)$ of this region $\hat{a}$ be a unit vector in any direction. If ∂s is a small distance from P in the direction $\hat{a}$, then

$$\frac{d}{ds}f(P) \quad \text{i.e.} \quad \lim_{\partial s \to 0} \frac{\partial f}{\partial s}$$

is called the directional derivative at the point P on the function f in the direction of $\hat{a}$.

If ∂s is taken in the direction of x-axis, then this will be ∂x.

Hence $\dfrac{\partial f}{\partial x}$ is the directional derivative of $f(x, y, z)$ is the direction of x-axis.

Similarly $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$ are the directional derivatives of $f(x, y, z)$ in the directions of y-axis and z-axis respectively.

Some theorem related to Directional Derivatives :

Theorem : The directional derivative of a scalar field f at a given point $P(x, y, z)$ in the direction of a unit vector $\hat{a}$ is given by :

$$\frac{\partial f}{\partial s} = \nabla f \cdot \hat{a} = (\text{grad } f) \cdot \hat{a}$$

Proof. Let $f(x, y, z)$ be a scalar point function in any region R and position vector of any point $P(x, y, z)$ in this region is $r = xi + yj + zk$. If the distance of the point P from a fixed point A in the direction of $\hat{a}$ be s, then δs will represent a small distance at P in the direction $\hat{a}$.

Hence $\dfrac{dr}{ds}$ will be a unit vector at point P in the direction of $\hat{a}$. Therefore $\dfrac{dr}{ds} = \hat{a}$

But $\quad r = xi + yj + zk$

$$\therefore \quad \frac{dr}{ds} = \frac{dx}{ds}i + \frac{dy}{dz}j + \frac{dz}{ds}k$$

or, $\quad \hat{a} = \dfrac{dx}{ds}i + \dfrac{dy}{ds}j + \dfrac{dz}{ds}k \qquad \qquad ...(1)$

Now, $\quad \nabla f \cdot \hat{a} = \left(i\dfrac{\partial f}{\partial x} + j\dfrac{\partial f}{\partial y} + k\dfrac{\partial f}{\partial z} \right)\left(\dfrac{dx}{ds}i + \dfrac{dy}{ds}j + \dfrac{dz}{ds}k \right) \qquad \text{[by (1)]}$

$$= \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} = \frac{df}{ds}$$

= the directional derivative of f at point P in the direction of â.

Theorem : ∇f (= grad f) is a vector normal to the surface f(x, y, z) = c where c is a constant.

Proof. Let P(x, y, z) be a point on the surface f(x, y, z) = c

and   Q(x + δx, y + δy, z + δz) be its neighbouring point on this surface.

Let   r = xi + yj + zk

and   r + δr = (x + δx)i + (y + δy)j + (z + δz)k

which are the position vectors of P and Q respectively.

Therefore $\overrightarrow{PQ} = \delta r = \delta x\, i + \delta y\, j + \delta z\, k$ ...(1)

When Q → P, then the line PQ tend to the tangent at the point P on the given surface. Hence in the limiting position, (1) becomes

dr = dx i + dy j + dz k ...(2)

which lies in the tangent plane at the point P on the surface.

Again by Differential Calculus,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= \left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}\right)\cdot\left(dx\, i + dy\, j + dz\, k\right)$$

$$= \nabla f \cdot dr \qquad\qquad \text{[by (2)]}$$

But   f(x, y, z) = constant $\Rightarrow$ df = 0

Hence $\nabla f.dr = 0 \Rightarrow \nabla f$ perpendicular to the vector dr.

$\Rightarrow$ ∇f is perpendicular to the tangent plane at P on the surface.

$$[\because \text{ dr lies in the tangent plane at P on the surface.}]$$

$\Rightarrow$ ∇ f i.e., grad f is normal to the surface f(x, y, z) = c.

Theorem : If n̂ be a unit vector normal to the level surface f(x, y, z) = c at a point P in the direction of f increasing and n be the distance of P from a fixed point A in the direction of n̂, then.

$$\text{grad } f = \frac{df}{dn}\hat{n}$$

Proof. By theorem grad f is normal to the plane f(x, y, z) = c, therefore

$$\text{grad } f = \lambda\hat{n} \qquad\qquad\qquad ...(1)$$

where λ is any constant.

Again by theorem

$$(\text{grad } f) \cdot n = \frac{\partial f}{\partial n} \qquad\qquad\qquad ...(2)$$

From (1) and (2),

$$(\lambda\hat{n})\cdot\hat{n} = \frac{\partial f}{\partial n} \Rightarrow \lambda = \frac{\partial f}{\partial n}$$

Substituting the value of λ in (1),

$$\text{grad } f = \frac{\partial f}{\partial n}\hat{n}$$

Remark : $|\text{grad } f| = |\nabla f| = \dfrac{\partial f}{\partial n}$

Theorem : grad f is a vector in the direction in which the maximum value of $\dfrac{df}{ds}$ (directional derivative) occurs.

Proof. We know that the directional derivative of f in the direction of $\hat{a}$

$$\frac{df}{ds} = (\text{grad } f)\cdot\hat{a} = \left(\frac{df}{dn}\hat{n}\right)\cdot\hat{a} \qquad\qquad\text{[by theorem]}$$

$$= \frac{\partial f}{\partial n}(\hat{n}\cdot\hat{a}) = \frac{\partial f}{\partial n}\cos\theta,$$

where θ is the angle between the vectors $\hat{a}$ and $\hat{n}$.

Now since $\dfrac{\partial f}{\partial n}$ is given, therefore the value of $\dfrac{df}{ds}$ depend on $\cos\theta$.

Hence the value of $\dfrac{df}{ds}$ will be maximum when $\theta = 0$ i.e. when $\hat{a}$ is in the direction of $\hat{n}$ i.e., when $\hat{a}$ is in the direction of the normal to f.

i.e. maximum directional derivative of f is in the direction of the normal to the plane and the **value of the maximum directional derivative.**

$$= \frac{df}{dn} = |\text{grad } f|.$$

Vector equation of the Tangent plane :

The vector equation of the tangent plane to the surface f(x, y, z) = c.

Let P(a, b, c) be a point in the given plane whose position vector is

Therefore r_0 = ai + bj + ck

Let Q(x, y, z) be a point on the tangent plane at P whose position vector is r, therefore

$\quad$ r = xi + yj + zk

then $\quad \overline{OP} = r - r_0 = (x-a)i + (y-b)j + (z-c)k$

Now since

$$\nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$$

is the direction of the normal to the plane and $\overline{PQ}$ is perpendicular to ∇f being in the tangent plane.

$$(r - r_0)\cdot\nabla f = 0$$

which is the vector equation of the tangent plane to the surface at the point P.

Vector Equation of the Normal

The vector equation of the normal to the surface f(x, y, z) = c.

Take Q(x, y, z) any point on the normal at the point P whose position vector is r, then $\overline{PQ}$ and ∇f will be parallel, PQ being normal to the surface

∴ $\quad (r - r_0) \times \nabla f = 0$

which is the vector equation of the normal to the surface at P.

Example : Find the directional derivative of f = xy + yz + zx in the direction of the vector i + 2j + 2k at the point (1, 2, 0)

Solution. Here ϕ = xy + yz + zx

∴ $\quad \nabla\phi = i\dfrac{\partial\phi}{\partial x} + j\dfrac{\partial f}{\partial y} + k\dfrac{\partial f}{\partial z}$

$\quad = i\dfrac{\partial}{\partial x}(xy + yz + zx) + j\dfrac{\partial}{\partial y}(xy + yz + zx) + k\dfrac{\partial}{\partial z}(xy + yz + zx)$

$\quad = i(y + z) + j(z + x) + k(x + y) = (y + z)i + (z + x)j + (x + y)k$

∴ At the point (1, 2, 0).

$\quad \nabla\phi = (2 + 0)i + (0 + 1)j + (1 + 2)k = 2i + j + 3k$ ...(1)

Again if $\hat{a}$ is the unit vector in the direction of the vector i + 2j + 2k

$\quad \hat{a} = \dfrac{i + 2j + 2k}{\sqrt{(1 + 4 + 4)}} = \dfrac{1}{3}(i + 2i + 2k)$ ...(2)

Therefore the required directional derivative = $(\nabla\phi)\cdot\hat{a}$

$\quad = (2i + j + 3k)\cdot\dfrac{1}{3}(i + 2j + 2k)$

$\quad = \dfrac{1}{3}(2 + 2 + 6)$ $\qquad$ [(by (1) and (2)]

$\quad = \dfrac{10}{3}$

Example : For the function f = $\dfrac{y}{x^2 + y^2}$; find the magnitude of the directional derivative making an angle 30° with the positive x-axis at the point (0, 1).

Solution. f = y/(x² + y²)

∴ $\quad \nabla f = i\dfrac{\partial f}{\partial x} + j\dfrac{\partial f}{\partial y} + k\dfrac{\partial f}{\partial z}$

$\quad = i\dfrac{\partial f}{\partial x}\left(\dfrac{y}{x^2 + y^2}\right) + j\dfrac{\partial f}{\partial y}\left(\dfrac{y}{x^2 + y^2}\right) + k\dfrac{\partial f}{\partial z}\left(\dfrac{y}{x^2 + y^2}\right)$

$\quad = i\left[\dfrac{-2xy}{\left(x^2 + y^2\right)^2}\right] + j\left[\dfrac{x^2 - y^2}{\left(x^2 + y^2\right)^2}\right]$

Therefore at the point (0, 1), ∇f = i(0) + j(−1) = −j ...(1)

Again if $\hat{a}$ is the unit vector which makes an angle 30° with the positive direction to x-axis at the point (0, 1), then

$$\hat{a} = (\cos 30°)i + (\sin 30°)j = \frac{\sqrt{3}}{2}i + \frac{1}{2}j$$

∴   Required directional derivative = $(\nabla f) \cdot \hat{a}$

$$= (-j) \cdot \left(\frac{\sqrt{3}}{2}i + \frac{1}{2}j \right)$$ [by (1) and (2)]

$$= -\frac{1}{2}$$

Example : Find the angle between the surface $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2, −1, 2).

Solution. Let $f(x, y, z) \equiv x^2 + y^2 + z^2 - 9 = 0$

$\phi(x, y, z) \equiv x^2 + y^2 - z - 3 = 0$

∴   grad f = $\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$

$$= 2xi + 2yj + 2zk$$

Therefore at the point (2, −1, 2), grad f = 4i − 2j + 4k   ...(1)

Again grad ϕ = $\nabla \phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$

$$= 2xi + 2yj - k$$

∴   At the (2, −1, 2), grad f = 4i − 2j − k   ...(2)

Now since grad f is in the direction of the normal tot he surface f(x, y, z), therefore at the point (2, −1, 2), the angle between the surfaces f(x, y, z) and ϕ(x, y, z) is the angle between grad f and grad ϕ. If this angle is θ, then

$$\cos \theta = \frac{(\text{grad } f) \cdot (\text{grad } \phi)}{|\text{grad } f||\text{grad } \phi|} = \frac{(4i - 2j + 4k) \cdot (4i - 2j - k)}{\sqrt{(16 + 4 + 16)}\sqrt{(16 + 4 + 1)}}$$

$$= \frac{16 + 4 - 4}{6\sqrt{21}} = \frac{8\sqrt{21}}{63}$$

Therefore the required angle = $\cos^{-1}\left(\frac{8\sqrt{21}}{63} \right)$

Example : Find the direction and magnitude of maximum directional derivative of $f = x^2 y z^3$ at the point (2, 1, −1).

Solution. The directional derivative of any scalar point function at any point is maximum in the direction of the gradient.

But   grad f = $\nabla f = i\frac{\partial}{\partial x}\left(x^2yz^3\right) + j\frac{\partial}{\partial y}\left(x^2yz^3\right) + k\frac{\partial}{\partial z}\left(x^2yz^3\right)$

$$= (2xyz^3)i + (x^2z^3)j + (3x^2y\,z^2)k$$

∴   At the point (2, 1, −1), grad f = −4i − 4j + 12 k

Therefore the direction of the maximum directional derivative at the point (2, 1, −1)

$$= -4i - 4j + 12k$$

Again the magnitude of the directional derivative at the point (2, 1, −1)

= |grad f|

$$= \sqrt{(16+16+144)} = 4\sqrt{11}$$

Example : Find a unit vector normal to the surface $x^2 y + 2xz = 4$ at the point (2, −2, 3).

Solution. Let $f(x, y, z) \equiv x^2 y + 2xz - 4$

We know that grad f is a vector in the direction of normal at the point (x, y, z) to the surface f(x, y, z) = c.

$$\text{grad } f = \nabla f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right)\left(x^2 y + 2xz - 4 \right)$$

$$= (2xy + 2z)i + x^2 j + 2xk$$

∴ At point (2, −2, 3), grad f = −2i + 4j + 4k

Therefore −2i + 4j + 4k is a vector in the direction of normal at the point (2, −2, 3) on the given surface. Therefore unit vector at the point (2, −2, 3) normal to the surface.

$$= \frac{-2i + 4j + 4k}{\sqrt{(4+16+16)}} = \frac{1}{3}(-i + 2j + 2k)$$

Example : Find the equations of the tangent plane and the normal to the surface xyz = 4 at the point (1, 2, 2).

Solution. The given surface f(x, y, z) = xyz − 4

Let co-ordinates of any point on the surface f be (x, y, z) and position vector be r. If r_0 be the position vector of the point (1, 2, 2), then

$$r - r_0 = (xi + yj + zk) - (i + 2j + 2k)$$

$$= (x - 1)i + (y - 2)j + (z - 2)k$$

Again grad $f = \nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = yz\, i + xz\, j + xy\, k$

At the point (1, 2, 2), grad f = 4i + 2j + 2k

∴ The equation of the tangent plane at (1, 2, 2)

$(r - r_0)\cdot\text{grad } f = 0$

⇒ $\{(x - 1)i + (y - 2)j + (z - 2)k\} \cdot (4i + 2j + 2k) = 0$

⇒ $4(x - 1) + 2(y - 2) + 2(z - 2) = 0$

⇒ $2x + y + z = 6$

Again equation of the normal at the point (1, 2, 2)

$(r - r_0) \times \text{grad } f = 0$

⇒ $\{(x - 1)i + (y - 2)j + (z - 2)k\} \times (4i + 2j + 2k) = 0$

⇒ $\begin{vmatrix} i & j & k \\ x-1 & y-2 & z-2 \\ 4 & 2 & 2 \end{vmatrix} = 0$

⇒ $(y - z)i + (2z - x - 3)j + (x - 2y + 3)k = 0$

⇒ $y - z = 0 = 2z - x - 3 = x - 2y + 3$

⇒ $\frac{x + 3}{2} = y = z$

4. DIVERGENCE OF A VECTOR POINT FUNCTION

Definition : If f(x, y, z) is a continuous and differentiable vector point function, then the divergence of f is expressed as div f or $\nabla \cdot f$ and defined as

$$\text{div } f = \nabla \cdot f = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot f$$

$$= i \cdot \frac{\partial f}{\partial x} + j \cdot \frac{\partial f}{\partial y} + k \cdot \frac{\partial f}{\partial z}$$

Therefore $\nabla \cdot f = \sum i \cdot \frac{\partial f}{\partial x}$

From the definition it is clear that the divergence of the vector point function is a scalar point function.

If the divergence of any vector f is zero i.e., if

$$\text{div } f = \nabla \cdot f = 0$$

then this is called solenoidal vector.

Remark : $\nabla \cdot f \neq f \cdot \nabla$, because $f \cdot \nabla$ is an operator and not a vector.

Theorems on Divergence

Theorem : If f(x, y, z) be a continuous differentiable vector point function and $f = f_x i + f_y j + f_z k$,

then div f = $\nabla \cdot f = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$.

Proof. $\because$ $f = f_x i + f_y j + f_z k$, Therefore by definition,

$$\text{div } f = \nabla \cdot f = i \cdot \frac{\partial f}{\partial x} + j \cdot \frac{\partial f}{\partial y} + k \cdot \frac{\partial f}{\partial z}$$

$$= \sum i \cdot \frac{\partial f}{\partial x} = \sum i \cdot \frac{\partial}{\partial x} \left(f_x i + f_y j + f_z k \right)$$

$$= \sum i \cdot \left[\frac{\partial f_x}{\partial x} i + \frac{\partial f_y}{\partial x} j + \frac{\partial f_z}{\partial x} k \right] = \sum \frac{\partial f_x}{\partial x}$$

$$= \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

Theorem : If f and g be two differentiable vector point functions;

then   div(f + g) = div f + div g

i.e., $\nabla \cdot (f + g) = \nabla \cdot f + \nabla \cdot g$

Proof. div(f + g) = $\nabla \cdot (f + g)$

$$= i \cdot \frac{\partial}{\partial x} (f + g) + j \cdot \frac{\partial}{\partial y} (f + g) + k \cdot \frac{\partial}{\partial z} (f + g)$$

$$= i \cdot \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x} \right) + j \cdot \left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y} \right) + k \cdot \left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial z} \right)$$

$$=\left(i\cdot\frac{\partial f}{\partial x}+j\cdot\frac{\partial f}{\partial y}+k\cdot\frac{\partial f}{\partial z}\right)+\left(i\cdot\frac{\partial g}{\partial x}+j\cdot\frac{\partial g}{\partial y}+k\cdot\frac{\partial g}{\partial z}\right)$$

$$=\nabla\cdot f+\nabla\cdot g$$

$$=\text{div } f+\text{div } g$$

Theorem: If a is constant vector, then div a = 0.

Proof. ∵ a is a constant vector,

∴ $\frac{\partial a}{\partial x}=0,\frac{\partial a}{\partial y}=0,\frac{\partial a}{\partial z}=0$

Hence div a = $\nabla\cdot a = i\cdot\frac{\partial a}{\partial x}+j\cdot\frac{\partial a}{\partial y}+k\cdot\frac{\partial a}{\partial z}$

$$= 0+0+0 = 0$$

Example : If f = xy²i + 2x² yz j – 3yz²k ; find div f at the point (1, –1, 1).

Solution. div f = $\nabla\cdot f$

$$=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot\left(xy^2 i+2x^2 yz j-3yz^2 k\right)$$

$$=\frac{\partial}{\partial x}\left(xy^2\right)+\frac{\partial}{\partial y}\left(2x^2 yz\right)+\frac{\partial}{\partial z}\left(-3yz^2\right)$$

$$= y^2+2x^2 z-6yz$$

∴ At the point (1, –1, 1), div f = 1 + 2 + 6 = 9.

Example : If f = xy sin zi + y² sin x j + z² sin xy k,

then find $\nabla\cdot f$ at $\left(0,\frac{1}{2}\pi,\frac{1}{2}\pi\right)$.

Solution. $\nabla\cdot f=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot\left(xy\sin zi+y^2\sin x j+z^2\sin xy k\right)$

$$=\frac{\partial}{\partial x}(xy\sin z)+\frac{\partial}{\partial y}(y^2\sin x)+\frac{\partial}{\partial z}(z^2\sin xy)$$

$$= y\sin z+2y\sin x+2z\sin xy$$

∴ At the point $\left(0,\frac{1}{2}\pi,\frac{1}{2}\pi\right)$, $\nabla\cdot f=\frac{\pi}{2}+0+0=\frac{\pi}{2}$

Example : If f = (ax + 3y + 4z)i + (x – 2y + 3z)i + (3x + 2y – z)k is a solenoidal vector, find a.

Solution. Since f is a solenoidal vector, therefore $\nabla\cdot f=0$

⇒ $\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot f=0$

⇒ $\frac{\partial}{\partial x}(ax+3y+4z)+\frac{\partial}{\partial y}(x-2y+3z)+\frac{\partial}{\partial z}(3x+2x-z)=0$

⇒ a – 2 – 1 = 0 ⇒ a = 3

Example : If r = xi + yj + zk, prove that :

(a)   div r = 3   (b) $\text{div } \hat{r} = \dfrac{2}{r}$ (c) $\text{div}\left(\dfrac{r}{r^3}\right) = 0$

Solution. (a) $\text{div } r = \nabla \cdot r = \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z}\right) \cdot (xi + yj + zk)$

$$= \dfrac{\partial x}{\partial x} + \dfrac{\partial y}{\partial y} + \dfrac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

(b) $\text{div } \hat{r} = \nabla \cdot \hat{r} = \nabla \cdot \dfrac{r}{r}$

$$= \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z}\right) \cdot \left(\dfrac{xi + yj + zk}{r}\right)$$

$$= \dfrac{\partial}{\partial x}\left(\dfrac{x}{r}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{y}{r}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{z}{r}\right)$$

$$= \dfrac{1}{r} - \dfrac{x}{r^2}\dfrac{\partial r}{\partial x} + \dfrac{1}{r} - \dfrac{y}{r^2}\dfrac{\partial r}{\partial y} + \dfrac{1}{r} - \dfrac{z}{r^2}\dfrac{\partial r}{\partial z}$$

$$= \dfrac{3}{r} - \dfrac{x}{r^2}\dfrac{x}{r} - \dfrac{y}{r^2}\dfrac{y}{r} - \dfrac{z}{r^2}\dfrac{z}{r} \qquad \left[\because \ \dfrac{\partial r}{\partial x} = \dfrac{x}{r},\ \dfrac{\partial r}{\partial y} = \dfrac{y}{r},\ \dfrac{\partial r}{\partial z} = \dfrac{z}{r}\right]$$

$$= \dfrac{3}{r} - \dfrac{1}{r^3}\left(x^2 + y^2 + z^2\right)$$

$$= \dfrac{3}{r} - \dfrac{1}{r^3}\left(r^2\right) = \dfrac{2}{r}$$

(c) $\text{div}\left(\dfrac{r}{r^3}\right) = \nabla \cdot \left(\dfrac{r}{r^3}\right)$

$$= \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z}\right) \cdot \left(\dfrac{xi + yj + zk}{r^3}\right)$$

$$= \dfrac{\partial}{\partial x}\left(\dfrac{x}{r^3}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{y}{r^3}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{z}{r^3}\right)$$

$$= \left[\dfrac{r}{r^3} - \dfrac{3x}{r^4}\dfrac{\partial r}{\partial x}\right] + \left[\dfrac{1}{r^3} - \dfrac{3y}{r^4}\dfrac{\partial r}{\partial y}\right] + \left[\dfrac{1}{r^3} - \dfrac{3z}{r^4}\dfrac{\partial r}{\partial z}\right]$$

$$= \dfrac{3}{r^3} - \dfrac{3x}{r^4}\dfrac{x}{r} - \dfrac{3y}{r^4}\dfrac{y}{r} - \dfrac{3z}{r^4}\dfrac{z}{r}$$

$$= \dfrac{3}{r^3} - \dfrac{3}{r^5}\left(x^2 + y^2 + z^2\right) = \dfrac{3}{r^3} - \dfrac{3}{r^5}\left(r^2\right) = 0$$

Example : If r = xi + yj + zk and r = |r|; prove that :

div $r^n r = (n + 3)r^n$

Hence show that $r^n r$ will be solenoidal, if $n = -3$

Solution. div $r^n r = \nabla \cdot r^n r$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot r^n \left(xi + yj + zk \right)$$

$$= \frac{\partial}{\partial x}\left(r^n x \right) + \frac{\partial}{\partial y}\left(r^n y \right) + \frac{\partial}{\partial z}\left(r^n z \right)$$

$$= \left(r^n \frac{\partial x}{\partial x} + xnr^{n-1}\frac{\partial r}{\partial x} \right) + \left(r^n \frac{\partial y}{\partial y} + ynr^{n-1}\frac{\partial r}{\partial y} \right) + \left(r^n \frac{\partial z}{dz} + znr^{n-1}\frac{\partial r}{\partial z} \right)$$

$$= \left(r^n + nxr^{n-1}\frac{x}{r} \right) + \left(r^n + nyr^{n-1}\frac{y}{r} \right) + \left(r^n + nzr^{n-1}\frac{z}{r} \right) \qquad \left[\because \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r} \right]$$

$$= 3r^n + nr^{n-2}\left(x^2 + y^2 + z^2 \right)$$

$$= 3r^n + nr^{n-2}\left(r^2 \right) = (n + 3)r^n$$

Again $r^n r$ will be solenoidal, if div $r^n r = 0$

$\Rightarrow \qquad (n + 3)r^n = 0$

$\Rightarrow \qquad n + 3 = 0 \qquad\qquad\qquad\qquad [\because\ r \neq 0]$

$\Rightarrow \qquad n = -3$

Example : If a is a constant vector and r = xi + yj + zk; prove that :

$\nabla \cdot (r \times a) = 0$

Solution. Let $a = a_1 i + a_2 j + a_3 k$, then

$$r \times a = \begin{vmatrix} i & j & k \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= i(a_3 y - a_2 z) + j(a_1 z - a_3 x) + k(a_2 x - a_1 y)$$

$\therefore \qquad \nabla \cdot (r \times a) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot [i(a_3 y - a_2 z) + j(a_1 z - a_3 x) + k(a_2 x - a_1 y)]$

$$= \frac{\partial}{\partial x}(a_3 y - a_2 z) + \frac{\partial}{\partial y}(a_1 z - a_3 x) + \frac{\partial}{\partial z}(a_2 x - a_1 y)$$

$$= 0 + 0 + 0 = 0$$

Aliter :

$$\text{div}(r \times a) = \Sigma i \cdot \frac{\partial}{\partial x}(r \times a) = \Sigma i \cdot \left[\frac{\partial r}{\partial x} \times a + r \times \frac{\partial a}{\partial x} \right]$$

$$= \Sigma i \cdot [i \times a + r \times 0] \qquad\qquad \left[\because \frac{\partial r}{\partial x} = i, \frac{\partial a}{\partial x} = 0 \right]$$

$$= \Sigma i \cdot (i \times a) = \Sigma [i\ i\ a]$$

$$= \Sigma\ 0 = 0$$

Example : If r = xi + yj + zk and r = |r|; prove that :

$$\text{div}\left(\frac{f(r)}{r}r\right) = \frac{1}{r^2}\frac{d}{dr}\left[r^2 f(r)\right]$$

Solution. $\text{div}\left(\dfrac{f(r)}{r}r\right) = \nabla \cdot \dfrac{f(r)}{r}r$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \frac{f(r)}{r}(xi + yj + zk)$$

$$= \frac{\partial}{\partial x}\left\{\frac{f(r)}{r}x\right\} + \frac{\partial}{\partial y}\left\{\frac{f(r)}{r}y\right\} + \frac{\partial}{\partial z}\left\{\frac{f(r)}{r}z\right\}$$

$$= \Sigma\frac{\partial}{\partial x}\left[\frac{f(r)}{r}x\right] = \Sigma\left[\frac{f(r)}{r}\frac{\partial x}{\partial x} + x\frac{\partial}{\partial x}\left\{\frac{f(r)}{r}\right\}\right]$$

$$= \Sigma\left[\frac{f(r)}{r} + x\frac{rf'(r)\frac{\partial r}{\partial x} - f(r)\frac{\partial r}{\partial x}}{r^2}\right]$$

$$= \Sigma\left[\frac{f(r)}{r} + \frac{x^2}{r^3}\left\{rf'(r) - f(r)\right\}\right]$$

$$= \frac{3f(r)}{r} + \frac{x^2 + y^2 + z^2}{r^3}\left\{rf'(r) - f(r)\right\}$$

$$= \frac{3f(r)}{r} + \frac{r^2}{r^3}\left\{rf'(r) - f(r)\right\} = \frac{2f(r)}{r} + f'(r)$$

$$= \frac{1}{r^2}\left[2r\,f(r) + r^2 f'(r)\right] = \frac{1}{r^2}\frac{d}{dr}\left[r^2 f(r)\right]$$

5. CURL OF A VECTOR POINT FUNCTION

Definition : If f(x, y, z) is a continuous and differentiable vector point function, then the curl of f is expressed as curl f or ∇ × f and defined as

$$\text{curl } f = \nabla \times f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \times f$$

$$= i \times \frac{\partial f}{\partial x} + j \times \frac{\partial f}{\partial y} + k \times \frac{\partial f}{\partial z}$$

Therefore $\nabla \times f = \nabla i \times \dfrac{\partial f}{\partial x}$

From the definition it is clear that the curl a vector point function is vector.

Irrotational Vector : If the curl of any vector f is zero vector, i.e. if

$$\text{curl } f = \nabla \times f = 0$$

Then this is called an irrotational vector.

Theorems on Curl

Theorem : If $f = f_1 i + f_2 j + f_3 k$ is a differential vector point function, then

$$\text{curl } f = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) i + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) j + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) k$$

Proof. By definition,

$$\text{curl } f = \nabla \times f = i \times \frac{\partial f}{\partial x} + j \times \frac{\partial f}{\partial y} + k \times \frac{\partial f}{\partial z}$$

$$= i \times \frac{\partial}{\partial x}(f_1 i + f_2 j + f_3 k) + j \times \frac{\partial}{\partial y}(f_1 i + f_2 j + f_3 k) + k \times \frac{\partial}{\partial z}(f_1 i + f_2 j + f_3 k)$$

$$= i \times \left(\frac{\partial f_1}{\partial x} i + \frac{\partial f_2}{\partial x} j + \frac{\partial f_3}{\partial x} k \right) + j \times \left(\frac{\partial f_1}{\partial y} i + \frac{\partial f_2}{\partial y} j + \frac{\partial f_3}{\partial y} k \right) + k \times \left(\frac{\partial f_1}{\partial z} i + \frac{\partial f_2}{\partial z} j + \frac{\partial f_3}{\partial z} k \right)$$

$$= \left(\frac{\partial f_2}{\partial x} k - \frac{\partial f_3}{\partial x} j \right) + \left(-\frac{\partial f}{\partial y} k + \frac{\partial f_3}{\partial y} i \right) + \left(\frac{\partial f_1}{\partial z} j - \frac{\partial f_2}{\partial z} i \right)$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) i + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) j + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) k$$

Remark : The above relation can be written in the determinant form as

$$\text{curl } f = \nabla \times f = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

It should be kept in mind that the operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ be written before the functions f_1, f_2, f_3.

Theorem : If f and g be two differentiable vector point functions, then :

curl(f + g) = curl f + curl g

i.e., $\nabla \times (f + g) = (\nabla \times f) + (\nabla + g)$

Proof. curl(f + g) = Σ i $\times \frac{\partial}{\partial x}(f + g) = \Sigma i \times \left[\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x} \right]$

$$= \Sigma i \times \frac{\partial f}{\partial x} + \Sigma i \times \frac{\partial g}{\partial x}$$

= curl f + curl g

$\therefore \quad \nabla(f + g) = (\nabla \times f) + (\nabla \times g)$

Theorem : If a is a constant vector, then :

curl a = $\nabla \times$ a = 0

Proof. $\because$ a is a constant vector,

$\therefore \quad \frac{\partial a}{\partial x} = 0, \frac{\partial a}{\partial y} = 0, \frac{\partial a}{\partial z} = 0$

Hence $\nabla \times$ a = i $\times \frac{\partial a}{\partial x} + j \times \frac{\partial a}{\partial y} + k \times \frac{\partial a}{\partial z}$

= 0 + 0 + 0 = 0

Example : If f = xy^2i + $2x^2yz$ j $-$ 3 yz^2k; find the value of curl at the point (1, $-$1, 1).

Solution. curl f = $\nabla \times$ f = $\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz) \right]i + \left[\frac{\partial}{\partial z}(xy^2) - \frac{\partial}{\partial x}(-3yz^2) \right]j + \left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2) \right]k$$

$$= (-3z^2 - 2x^2y)i + (4xyz - 2xy)k$$

$\therefore \quad$ At the point (1, $-$1, 1)

curl f = ($-$3 + 2)i + ($-$4 + 2)k = $-$i $-$ 2k.

Example : If f = (x + y + 1)i + j + ($-$ x $-$ y)k ;

Prove that : f $\cdot$ curl f = 0

Solution. curl f = $\nabla \times$ f = $\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y + 1 & 1 & -(x + y) \end{vmatrix}$

= ($-$1)i + (1)j + ($-$1)k = $-$i + j $-$ k

$\therefore \quad$ f $\cdot$ curl f = [(x + y + 1)i + j + ($-$ x $-$ y)k] $\cdot$ ($-$i + j $-$ k)

= (x + y + 1)($-$1) + (1)(1) + ($-$ x $-$ y)($-$1)

= $-$x $-$ y $-$ 1 + 1 + x + y = 0.

Example : If $V = \nabla(x^3 + y^3 + z^3 - 3xyz)$; find the following :

(a)   div V   (b)   curl V

Solution. $V = \nabla(x^3 + y^3 + z^3 - 3xyz)$

$$= \Sigma i \frac{\partial}{\partial x}\left(x^3 + y^3 + z^3 - 3xyz\right) = \Sigma i \left(3x^2 - 3yz\right)$$

$$= 3(x^2 - yz)i + 3(y^2 - zx)j + 3(z^2 - xy)k$$

(a)   div $V = \nabla \cdot V$

$$= \frac{\partial}{\partial x} 3(x^2 - yz) + \frac{\partial}{\partial y} 3(y^2 - zx) + \frac{\partial}{\partial z} 3(z^2 - xy)$$

$$= 6x + 6y + 6z = 6(x + y + z)$$

(b)   curl $V = \nabla \times V = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3\left(x^2 - yz\right) & 3\left(y^2 - zx\right) & 3\left(z^2 - xy\right) \end{vmatrix}$

$$= 3\Sigma \left[\frac{\partial}{\partial y}\left(z^2 - xy\right) - \frac{\partial}{\partial z}\left(y^2 - zx\right) \right] i$$

$$= 3\Sigma(-x + x)i = 3\Sigma 0 = 0$$

Example : If $f = e^{xyz} (i + j + k)$; find curl f.

Solution. curl $f = \nabla \times f = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{xyz} & e^{xyz} & e^{xyz} \end{vmatrix}$

$$= i\left[\frac{\partial}{\partial y}\left(e^{xyz}\right) - \frac{\partial}{\partial z}\left(e^{xyz}\right) \right] + j\left[\frac{\partial}{\partial z}\left(e^{xyz}\right) - \frac{\partial}{\partial x}\left(e^{xyz}\right) \right] + k\left[\frac{\partial}{\partial x}\left(e^{xyz}\right) - \frac{\partial}{\partial y}\left(e^{xyz}\right) \right]$$

$$= e^{xyz} [x(z - y)i + y(x - z)j + z(y - x)k]$$

Example : Prove that the following vector is irrational :

$$f = (\sin y + z)i + (x \cos y - z)j + (x - y)k$$

Solution. We know that curl $f = \nabla \times f = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin y + z & x \cos y - z & x - y \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(x - y) - \frac{\partial}{\partial z}(x \cos y - z) \right]i + \left[\frac{\partial}{\partial z}(\sin y + z) - \frac{\partial}{\partial x}(x - y) \right]j + \left[\frac{\partial}{\partial x}(x \cos y - z) - \frac{\partial}{\partial y}(\sin y + z) \right]k$$

$$= (-1 + 1)i + (1 - 1)i + (\cos y - \cos y)k$$

$$= 0$$

Therefore f is an irrotational vector.

Example : If $(xyz)^b (x^a i + y^a j + z^a k)$ is a irrotational vector, then prove that either b = 0 or a = −1.

Solution. Let f be the given vector, then

$$\text{curl } f = \nabla \times f = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^a(xyz)^b & y^a(xyz)^b & z^a(xyz)^b \end{vmatrix}$$

$$= \Sigma i\left[\frac{\partial}{\partial y}\left(x^b y^b z^{a+b} - \frac{\partial}{\partial z}\left(x^b y^{a+b} z^b\right)\right)\right]$$

$$= \Sigma i\left[b\, x^b y^{b-1} z^{a+b} - b\, x^b y^{a+b} z^{b-1}\right]$$

$$= b(xyz)^b \, \Sigma i(z^a y^{-1} - y^a z^{-1})$$

$$= b(xyz)^b \, \Sigma i\left(\frac{z^{a+1} - y^{a+1}}{yz}\right)$$

$$= b(xyz)^b\left[i\left(\frac{z^{a+1} - y^{a+1}}{yz}\right) + j\left(\frac{x^{a-1} - z^{a+1}}{zx}\right) + k\left(\frac{y^{a+1} - x^{a+1}}{xy}\right)\right]$$

Now f is irrotational vector $\Rightarrow$ curl f = 0

which is possible, when

$$b(xyz)^b = 0 \quad \text{or}, \quad z^{a+1} - y^{a+1} = 0 = x^{a+1} - z^{a+1} = y^{a+1} - x^{a+1}$$

i.e. when b = 0 or, $a + 1 = 0 \Rightarrow a = -1$

Example : If r = xi = yj + zk and a is any constant vector ; then prove that :

$$\text{curl}\frac{a \times r}{r^3} = -\frac{a}{r^3} + \frac{3(a \cdot r)}{r^5}r$$

Solution. Let $a = a_1 i + a_2 j + a_3 k$, then

$$\frac{a \times r}{r^3} = \frac{1}{r^3}\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \frac{1}{r^3}\left[(a_2 z - a_3 y)i + (a_3 x - a_1 z)j + (a_1 y - a_2 x)k\right]$$

$$\therefore \quad \text{curl}\frac{a \times r}{r^3} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{a_2 z - a_3 y}{r^3} & \dfrac{a_3 x - a_1 z}{r^3} & \dfrac{a_1 y - a_2 x}{r^3} \end{vmatrix}$$

$$= \Sigma i\left[\frac{\partial}{\partial y}\left(\frac{a_1 y - a_2 x}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{a_3 x - a_1 z}{r^3}\right)\right]$$

$$= \Sigma i\left[\frac{a_1}{r^3} - \frac{3(a_1 y - a_2 x)}{r^4}\frac{\partial r}{\partial y} + \frac{a_1}{r^3} + \frac{3(a_3 x - a_1 z)}{r^4}\frac{\partial r}{\partial z}\right]$$

$$= \Sigma i \left[\frac{2a_1}{r^3} - \frac{3(a_1 y - a_2 x)}{r^4} \frac{y}{r} + \frac{3(a_3 x - a_1 z)}{r^4} \frac{z}{r} \right]$$

$$= \Sigma i \left[\frac{2a_1}{r^3} + \frac{3}{r^5} \left(a_2 xy + a_1 y^2 + a_3 xz - a_1 z^2 \right) \right]$$

$$= \Sigma i \left[\frac{2a_1}{r^3} + \frac{3x(a_1 x + a_2 y + a_3 z)}{r^5} - \frac{3a_1(x^2 + y^2 + z^2)}{r^5} \right]$$

$$= \Sigma i \left[\frac{2a_1}{r^3} + \frac{3x(a \cdot r)}{r^5} - \frac{3a_1}{r^3} \right] = \Sigma i \left[\frac{3x(a \cdot r)}{r^5} - \frac{a_1}{r^3} \right]$$

$$= \frac{3}{r^5}(a \cdot r)(xi + yj + zk) - \frac{a_1 i + a_2 j + a_3 k}{r^3}$$

$$= \frac{3}{r^5}(a \cdot r)r - \frac{a}{r^3} = -\frac{a}{r^3} + \frac{3(a \cdot r)}{r^5} r$$

Some Important Vector identities

Identity I : div(u a) = u · div a + a · grad u

or, $\nabla \cdot (u\ a) = u(\nabla \cdot a) + a \cdot (\nabla\ u)$

$= u(\nabla \cdot a) + (a \cdot \nabla)u$

Proof. By definition of divergence,

$$div(u\ a) = i \cdot \frac{\partial}{\partial x}(u\ a) + j \cdot \frac{\partial}{\partial y}(u\ a) + k \cdot \frac{\partial}{\partial z}(u\ a)$$

$$= \Sigma i \cdot \frac{\partial}{\partial x}(ua) = \Sigma i \cdot \left[\frac{\partial u}{\partial x} a + u \frac{\partial a}{\partial x} \right]$$

$$= \Sigma \left[i \cdot \left(\frac{\partial u}{\partial x} a \right) \right] + \Sigma \left[i \cdot \left(u \frac{\partial a}{\partial x} \right) \right]$$

$$= \Sigma \left[\left(\frac{\partial u}{\partial x} i \right) \cdot a \right] + \Sigma \left[u \left(i \cdot \frac{\partial a}{\partial x} \right) \right] \qquad [\because a \cdot (nb) = (na) \cdot b = n(a \cdot b)]$$

$$= \left[\Sigma \frac{\partial u}{\partial x} i \right] \cdot a + u \Sigma \left[i \cdot \frac{\partial a}{\partial x} \right]$$

$= (\nabla u) \cdot a + u(\nabla \cdot a) = u(\nabla \cdot a) + a \cdot (\nabla u)$

Identity II : curl(u a) = (grad u) × a u curl a

or $\nabla \times (u\ a) = (\nabla\ u) \times a + u(\nabla \times a)$

Proof. curl(u a) = $\nabla \times (u\ a)$

$$= i \times \frac{\partial}{\partial x}(u\ a) + j \times \frac{\partial}{\partial y}(u\ a) + k \times \frac{\partial}{\partial z}(u\ a)$$

$$= \Sigma i \times \frac{\partial}{\partial x}(u a) = \Sigma \left[i \times \left(\frac{\partial u}{\partial x} a + u \frac{\partial a}{\partial x} \right) \right]$$

$$= \Sigma\left[i\times\left(\frac{\partial u}{\partial x}a\right)\right] + \Sigma\left[i\times\left(u\frac{\partial}{\partial x}\right)\right]$$

$$= \Sigma\left[\left(\frac{\partial u}{\partial x}i\right)\times a\right] + \Sigma u\left(i\times\frac{\partial a}{\partial x}\right) \qquad [\because a\times(nb)=(na)\times b=n(a\times b)]$$

$$= \left[\Sigma\frac{\partial u}{\partial x}i\right]\times a + u\Sigma\left(i\times\frac{\partial a}{\partial x}\right)$$

$$= (\nabla u)\times a + u(\nabla\times a)$$

Identity III : div (a × b) = b·curl a − a·curl b

or, $\nabla\cdot(a\times b) = b\cdot(\nabla\times a) - a\cdot(\nabla\times b)$

Proof. div (a × b) = $\nabla\cdot(a\times b)$

$$= i\cdot\frac{\partial}{\partial x}(a\times b) + j\cdot\frac{\partial}{\partial y}(a\times b) + k\cdot\frac{\partial}{\partial z}(a\times b)$$

$$= \Sigma\left[i\cdot\frac{\partial}{\partial x}(a\times b)\right] = \Sigma\left[i\cdot\left(\frac{\partial a}{\partial x}\times b + a\times\frac{\partial b}{\partial x}\right)\right]$$

$$= \Sigma\left[i\cdot\left(\frac{\partial a}{\partial x}\times b\right)\right] + \Sigma\left[i\cdot\left(a\times\frac{\partial b}{\partial x}\right)\right]$$

$$= \Sigma\left[\left(i\times\frac{\partial a}{\partial x}\right)\cdot b\right] - \Sigma\left[\left(i\times\frac{\partial b}{\partial x}\right)\cdot a\right]$$

$$= (\nabla\times a)\cdot b - (\nabla\times b)\cdot a$$

$$= b\cdot(\nabla\times a) - a\cdot(\nabla\times b)$$

Identity IV : curl (a × b) = (b·∇)a − b div a − (a·∇)b + a div b

or, $\nabla\times(a\times b) = (b\cdot\nabla)a - b(\nabla\cdot a) - (a\cdot\nabla)b + a(\nabla\cdot b)$

Proof. curl(a × b) = $\nabla\times(a\times b)$

$$= i\times\frac{\partial}{\partial x}(a\times b) + j\times\frac{\partial}{\partial y}(a\times b) + k\times\frac{\partial}{\partial z}(a\times b)$$

$$= \Sigma\left[i\times\frac{\partial}{\partial x}(a\times b)\right] = \Sigma\left[i\times\left(a\times\frac{\partial b}{\partial x} + \frac{\partial a}{\partial x}\times b\right)\right]$$

$$= \Sigma\left[i\times\left(a\times\frac{\partial b}{\partial x}\right)\right] + \Sigma\left[i\times\left(\frac{\partial a}{\partial x}\times b\right)\right]$$

$$= \Sigma\left[\left(i\cdot\frac{\partial b}{\partial x}\right)a - (i\cdot a)\frac{\partial b}{\partial x}\right] + \Sigma\left[(i\cdot b)\frac{\partial a}{\partial x} - \left(i\cdot\frac{\partial a}{\partial x}\right)b\right]$$

$$= \Sigma\left[\left(i\cdot\frac{\partial b}{\partial x}\right)a\right] - \Sigma\left[(a\cdot i)\frac{\partial b}{\partial x}\right] + \Sigma\left[(b\cdot i)\frac{\partial a}{\partial x}\right] - \Sigma\left[\left(i\cdot\frac{\partial a}{\partial x}\right)b\right]$$

$$= \left[\Sigma\left(i\cdot\frac{\partial b}{\partial x}\right)\right]a - \left[a\cdot\Sigma i\frac{\partial}{\partial x}\right]b + \left[b\cdot\Sigma i\frac{\partial}{\partial x}\right]a - \left[\Sigma\left(i\cdot\frac{\partial a}{\partial x}\right)\right]b$$

$$= (\nabla \cdot b)a - (a \cdot \nabla)b + (b \cdot \nabla)a - (\nabla \cdot a)b$$

$$= (b \cdot \nabla)a - b(\nabla \cdot a) - (a \cdot \nabla)b + a(\nabla \cdot a)$$

Particularly, when a is a constant vector, then

$$\nabla \times (a \times b) = a(\nabla \cdot b) - (a \cdot \nabla)b$$

Identity V : grad(a·b) = (b·∇)a + (a·∇)b + b × curl a + a × curl b

or,　　$\nabla(a \cdot b) = (b \cdot \nabla)a + (a \cdot \nabla)b + b \times (\nabla \times a) + a \times (\nabla \times b)$

Proof. grad(a·b) = ∇(a·b)

$$= i\frac{\partial}{\partial x}(a \cdot b) + j\frac{\partial}{\partial y}(a \cdot b) + k\frac{\partial}{\partial z}(a \cdot b)$$

$$= \Sigma i\frac{\partial}{\partial x}(a \cdot b) = \Sigma \left[i \left(a \cdot \frac{\partial b}{\partial x} + \frac{\partial a}{\partial x} \cdot b \right) \right]$$

$$= \Sigma \left[\left(a \cdot \frac{\partial b}{\partial x} \right) i \right] + \Sigma \left[\left(b \cdot \frac{\partial a}{\partial x} \right) i \right] \qquad \ldots(1)$$

∴ By vector triple product, a × (b × c) = (a·c)b − (a·b)c

$\Rightarrow$ 　(a·b)c = (a·c)b − a × (b × c)

∴ 　$\left(a \cdot \frac{\partial b}{\partial x} \right) i = (a \cdot i)\frac{\partial b}{\partial x} - a \times \frac{\partial b}{\partial x} \times i$

$$= (a \cdot i)\frac{\partial b}{\partial x} + a \times \left(i \times \frac{\partial b}{\partial x} \right)$$

$\Rightarrow$ 　$\Sigma \left[\left(a \cdot \frac{\partial b}{\partial x} \right) i \right] = \Sigma \left[(a \cdot i)\frac{\partial b}{\partial x} \right] + \Sigma \left[a \times \left(i \times \frac{\partial b}{\partial x} \right) \right]$

$$= \left[a \cdot \Sigma i \frac{\partial}{\partial x} \right] b + a \times \Sigma \left(i \times \frac{\partial b}{\partial x} \right)$$

$$= (a \cdot \nabla)b + a \times (\nabla \times b) \qquad \ldots(2)$$

Similarly, 　$\Sigma \left[\left(b \cdot \frac{\partial a}{\partial x} \right) i \right] = (b \cdot \nabla)a + b \times (\nabla \times a)$ 　　$\ldots(3)$

Substituting the values from (2) and (3) in (1),

grad(a·b) = (a·∇)b + a × (∇ × b) × (b·∇)a + b × (∇ × a)

$$= (b \cdot \nabla)a + (a \cdot \nabla)b + b \times (\nabla \times a) + a \times (\nabla \times b)$$

Second order differential function

We know that for any scalar point function φ, grad φ is a vector point function and for any vector point function f, div f is a scalar point function and curl f is a vector point function. Now since gradφ and curl f are vector functions, therefore their divergence and curl may exist.

Similarly, div f is a scalar function therefore gradient can be determined. Thus we can find (div(grad φ), curl(grad φ), div(curl f), curl(curl f) and grad(div f) which are called double order differentiable functions.

Property 1: div grad φ = ∇²φ

Proof. div gradφ = ∇·(∇φ)

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial \phi}{\partial x}i + \frac{\partial \phi}{\partial y}j + \frac{\partial \phi}{\partial z}k \right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$$= \nabla^2 \phi$$

Remark : Laplacian operator ∇^2 :

The operator $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

is called **Laplacian operator** and is equal to equal to $\nabla \cdot \nabla$ by property 1.

$$\nabla^2 \phi = (\nabla \cdot \nabla)\phi = \nabla \cdot (\nabla \phi)$$

This can be easily seen that

$$\nabla^2 \phi = 0 \Rightarrow \nabla^2 \phi_x = 0, \ \nabla^2 \phi_y = 0, \ \nabla^2 \phi_z = 0$$

Property 2 : curl grad u = 0 = $\nabla \times (\nabla\ u)$

i.e. the curl of the gradient of a scalar function u is zero.

Proof. $\because$ grad u = ∇u = $i\dfrac{\partial u}{\partial x} + j\dfrac{\partial u}{\partial y} + k\dfrac{\partial u}{\partial z}$

$\therefore$ curl grad u = $\nabla \times (\nabla u)$

$$= \left(i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}\right) \times \left(i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}\right)$$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2 u}{\partial z \partial y}\right)i + \left(\frac{\partial^2 u}{\partial z \partial x} - \frac{\partial^2 u}{\partial x \partial z}\right)j + \left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x}\right)k$$

$$= 0i + 0j + 0k = 0$$

Property 3: div curl a = 0 = $\nabla \cdot (\nabla \times a)$

i.e. the divergence of curl of any vector a is zero.

Proof. Let a = $a_1 i + a_2 j + a_3 k$, then

$$\text{curl a} = \nabla \times a = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right)i + \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right)j + \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)k$$

$\therefore$ div curl a = $\nabla \cdot (\nabla \times a) = \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z}\right) \cdot (\nabla \times a)$

$$= \Sigma \frac{\partial}{\partial x}\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) = \Sigma \left(\frac{\partial^2 a_3}{\partial x \partial y} - \frac{\partial^2 a_2}{\partial x \partial z}\right)$$

$$= \frac{\partial^2 a_3}{\partial x \partial y} - \frac{\partial^2 a_2}{\partial x \partial z} + \frac{\partial^2 a_1}{\partial y \partial z} - \frac{\partial^2 a_3}{\partial y \partial x} + \frac{\partial^2 a_2}{\partial z \partial x} - \frac{\partial^2 a_1}{\partial z \partial y} = 0$$

Property 4 : curl curl a = grad div a $- \Sigma \dfrac{\partial^2 a}{\partial x^2}$

i.e., $\quad \nabla \times (\nabla \times a) = \nabla(\nabla \cdot a) - \nabla^2 a$

Proof. Let a = $a_1 i + a_2 j + a_3 k$, then

$$\nabla \times a = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right)i + \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right)j + \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)k$$

$$\therefore \quad \nabla \times (\nabla \times a) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial a_3}{\partial y} - \dfrac{\partial a_2}{\partial z} & \dfrac{\partial a_1}{\partial z} - \dfrac{\partial a_3}{\partial x} & \dfrac{\partial a_2}{\partial x} - \dfrac{\partial a_1}{\partial y} \end{vmatrix}$$

$$= \Sigma\left[\frac{\partial}{\partial y}\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right)\right]i$$

$$= \Sigma\left[\frac{\partial^2 a_2}{\partial y \partial x} - \frac{\partial^2 a_1}{\partial y^2} - \frac{\partial^2 a_1}{\partial z^2} + \frac{\partial^2 a_3}{\partial z \partial x}\right]i$$

$$= \Sigma\left[\left(\frac{\partial^2 a_2}{\partial y \partial x} + \frac{\partial^2 a_3}{\partial z \partial x}\right) - \left(\frac{\partial^2 a_1}{\partial y^2} + \frac{\partial^2 a_1}{\partial z^2}\right)\right]i$$

$$= \Sigma\left[\frac{\partial}{\partial x}\left(\frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}\right) - \left(\frac{\partial^2 a_1}{\partial y^2} + \frac{\partial^2 a_1}{\partial z^2}\right)\right]i$$

$$= \Sigma\left[\frac{\partial}{\partial x}\left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}\right) - \left(\frac{\partial^2 a_1}{\partial x^2} + \frac{\partial^2 a_1}{\partial y^2} + \frac{\partial^2 a_1}{\partial z^2}\right)\right]i$$

$$= \Sigma\left[\frac{\partial}{\partial x}(\nabla \cdot a) - (\nabla^2 a_1)\right]i = \Sigma\left[i\frac{\partial}{\partial x}(\nabla \cdot a)\right] - \nabla^2 \Sigma a_1 i$$

$$= \nabla(\nabla \cdot a) - \nabla^2 a$$

Example : Prove that :

$$(a \cdot \nabla)a = \frac{1}{2}\nabla a^2 - a \times (\nabla \times a), \text{ where } a^2 = |a|^2$$

Solution. $a \times (\nabla \times a) = a \times \sum \left(i \times \dfrac{\partial a}{\partial x} \right)$

$$= \sum \left[a \times \left(i \times \dfrac{\partial a}{\partial x} \right) \right] = \sum \left[\left(a \cdot \dfrac{\partial a}{\partial x} \right) i - (a \cdot i) \dfrac{\partial a}{\partial x} \right]$$

$$= \sum \left[\dfrac{\partial}{\partial x} \left(\dfrac{1}{2} |a|^2 \right) \right] i - \sum (a \cdot i) \dfrac{\partial a}{\partial x}$$

$$= \dfrac{1}{2} \sum i \dfrac{\partial}{\partial x} |a|^2 - \left[\sum (a \cdot i) \dfrac{\partial}{\partial x} \right] a$$

$$= \dfrac{1}{2} \nabla |a|^2 - (a \cdot \nabla) a$$

$\therefore \qquad (a \cdot \nabla) a = \dfrac{1}{2} \nabla a^2 - a \times (\nabla \times a)$ (By transposition)

Integration of Vectors

Let f(t) be a finite, single valued and continuous vector function of the single scalar variable t, then the vector function of t, which when differentiated wrt t gives f(t) i.e. a vector function F(t) exist such that its differentiation wrt t.

$$\dfrac{d}{dt} [F(t)] = f(t)$$

then F(t) is called the *Integral* of vector function f(t) wrt t.

In the *symbolic form*,

if $\qquad \dfrac{dF(t)}{dt} = f(t)$

then $\qquad \int f(t) dt = F(t),$ ...(1)

Here the vector function f(t) is called the *Integrand*.

Vector constant of integration :

If $\qquad \dfrac{d}{dt} F(t) = f(t),$ then $\qquad \int f(t) dt = F(t) + c,$

where the constant vector c is called the arbitrary constant vector of integration.

Proof. $\dfrac{d}{dx} F(t) = f(t)$ ...(1)

$$\dfrac{d}{dx} [F(t) + c] = \dfrac{d}{dt} F(t) + \dfrac{d}{dt} (c)$$

$$= \dfrac{d}{dt} F(t) + 0 \qquad\qquad [\because \ c \text{ is a constant vector}]$$

$$= f(t) \qquad\qquad\qquad\qquad [\text{by (1)}]$$

Therefore by definition,

$$\int f(t) dt = F(t) + c,$$

where the constant vector c is called the arbitrary constant vector of integration.

This constant vector of integration is determined by the initial conditions or geometrical conditions.

Indefinite Integral : Since in F(t) + c, c is an indefinite constant vector, therefore this is called an indefinite integral of f(t) wrt t.

Theorem : If $f(t) = f_1(t)i + f_2(t)j + f_3(t)k$ is a vector function of single function t, where $f_1(t)$, $f_2(t)$ and $f_3(t)$ are continuous in a particular region,

then $\int f(t)dt = i\int f_1(t)dt + j\int f_2(t)dt + k\int f_3(t)dt$

Proof. Let $F(t) = i\int f_1(t)dt + j\int f_2(t)dt + k\int f_3(t)dt$

such that $\dfrac{d}{dt}F(t) = f(t)$ ...(1)

then $F(t) = \int f(t)dt$ ...(2)

Now by (1),

$$\dfrac{d}{dt}\{F_1(t)i + F_2(t)j + F_3(t)k\} = f_1(t)i + f_2(t)j + f_3(t)k$$

$$\dfrac{d}{dt}F_1(t)i + \dfrac{d}{dt}F_2(t)j + \dfrac{d}{dt}F_3(t)k = f_1(t)i + f_2(t)j + f_3(t)k$$

Comparing the coefficients of i, j and k,

$$\dfrac{d}{dt}F_1(t) = f_1(t); \quad \dfrac{d}{dt}F_2(t) = f_2(t); \quad \dfrac{d}{dt}F_3(t) = f_3(t)$$

$\therefore$ $F_1(t) = \int f_1(t)dt; \quad F_2(t) = \int f_2(t)dt; \quad F_3(t) = \int f_3(t)dt$

Therefore by (2),

$$\int F_1(t)dt\, i + \int f_2(t)dt\, j + \int f_3(t)dt\, k = \int f(t)dt$$

Hence to integrate a vector function, its components are integrated.

Some important results on integration

If f(t) and ϕ(t) are two continuous differential vector functions of scalar variable t expressed by f and ϕ, then with the help of Differential Calculus, their corresponding integrals can also be easily derived :

[I_1] $\int\left(f \cdot \dfrac{d\phi}{dt} + \phi \cdot \dfrac{df}{dt}\right)dt = f \cdot \phi + c$

By differentiation, we have

$$\dfrac{d}{dt}(f \cdot \phi) = f \cdot \dfrac{d\phi}{dt} + \phi \cdot \dfrac{df}{dt}$$

Now by definition, the required result is obtained.

[I_2] $\int\left(2f \cdot \dfrac{df}{dt}\right)dt = f^2 + c$

Taking f = ϕ in I_1 ;

$$\int\left(f\cdot\frac{df}{dt}+f\cdot\frac{df}{dt}\right)dt=f\cdot f+c$$

which gives the required result.

$[I_3]$ $\int 2\frac{df}{dt}\cdot\frac{d^2f}{dt^2}dt=\left(\frac{df}{dt}\right)^2+c$

Replacing f by $\frac{df}{dt}$ in I_2,

$$\int 2\frac{df}{dt}\cdot\frac{d^2f}{dt^2}dt=\frac{df}{dt}\cdot\frac{df}{dt}+c=\left(\frac{df}{dt}\right)^2+c$$

$[I_4]$ $\int\left(f\times\frac{d\phi}{dt}+\frac{df}{dt}\times\phi\right)dt=f\times\phi+c$

By differentiation, we have

$$\frac{d}{dt}\left(f\times\phi\right)=f\times\frac{d\phi}{dt}+\frac{df}{dt}\times\phi$$

Therefore by definition, $\int\left(f\times\frac{d\phi}{dt}+\frac{df}{dt}\times\phi\right)dt=f\times\phi+c$

$[I_5]$ $\int\left(a\times\frac{df}{dt}\right)dt=a\times f+c$, where a is a constant vector.

If a is a constant vector, then by differentiation,

$$\frac{d}{dt}\left(a\times f\right)=\frac{da}{dt}\times f+a\times\frac{df}{dt}$$

$$=a\times\frac{df}{dt}\qquad\qquad\left[\because\ \frac{da}{dt}=0\right]$$

Now by definition $\int\left(a\times\frac{df}{dt}\right)dt=a\times f\times c$

$[I_6]$ $\int\left(f\times\frac{d^2f}{dt^2}\right)dt=f\times\frac{df}{dt}+c$

By differentiation, we have

$$\frac{d}{dt}\left(f\times\frac{df}{dt}\right)=\frac{df}{dt}\times\frac{df}{dt}+f\times\frac{d^2f}{dt^2}$$

$$=f\times\frac{d^2f}{dt^2}\qquad\qquad\left[\because\ \frac{df}{dt}\times\frac{df}{dt}=0\right]$$

Now by differentiation, $\int\left(f\times\frac{d^2f}{dt^2}\right)dt=f\times\frac{df}{dt}+c$

[I_7] $\int\left(\dfrac{1}{a}\dfrac{da}{dt}-\dfrac{1}{a^2}\dfrac{da}{dt}a\right)dt=\hat{a}+c,$

where $\hat{a}$ is a unit vector in the direction of a and |a| = a.

We know that $\hat{a}=\dfrac{a}{a}$

Therefore $\dfrac{d}{dt}(\hat{a})=\dfrac{d}{dt}\left(\dfrac{a}{a}\right)=\dfrac{1}{a}\dfrac{da}{dt}-a^2\dfrac{da}{dt}a$

Now by definition,

$$\int\left(\dfrac{1}{a}\dfrac{da}{dt}-\dfrac{1}{a^2}a\dfrac{da}{dt}\right)dt=\hat{a}+c$$

Remark : If the *integrand* is scalar (dot product) then the constant of integration is a scalar and if the *integrand* is vector (cross product) then the constant of integration is a vector.

Definite Integral : Definition :

If $\int f(t)dt=F(t),$ then $F(b)-F(a)$

is called definite integral of f(t) between the limits t = a and t = b.

and is written as $\int_a^b f(t)dt$

i.e., $\int_a^b f(t)dt=\left[F(t)\right]_a^b=F(b)-F(a).$

Example : Find : $\int f(t)dt$, where f(t) = (t − t²)i + 2t³ j − 3k

Solution. Here the integral $\int\left\{(t-t^2)i+2t^3j-3k\right\}dt$

$=i\int(t-t^2)dt+j\int 2t^3\,dt-k\int 3\,dt$

$=\left(\dfrac{1}{2}t^2-\dfrac{1}{3}t^3\right)i+\dfrac{1}{2}t^4j-3t\,k+c,$

where c = c_1i + c_2j + c_3k.

Example : If r(t) = t i − t² j + (t − 1)k and s(t) = 2t² i + 6t k, then find the value of :

(a) $\int_0^2(r\cdot s)dt$ (b) $\int_0^2(r\times s)dt$

Solution. (a) r · s = {t i − t² j + (t − 1)k} · {2t² i + 6t k} = 2t - 2t² - 21 + 2t² + 12t

   = 2t³ + 6t² − 6t

Therefore $\int_0^2(r\cdot s)dt=\int_0^2(2t^3+6t^2-6t)dt$

$=\left[\dfrac{1}{2}t^4+2t^3-3t^2\right]_0^2=8+16-12=12$

(b)
$$r \times s = \{t\,i - t^2\,j + (t-1)k\} \times \{2t^2\,i + 6t\,k\} = \begin{vmatrix} i & j & k \\ t & -t^2 & t-1 \\ 2t^2 & 0 & 6t \end{vmatrix}$$

$$= \{-6t^3\,i + 2t^2(t-4)j + 2t^4\,k\}$$

Therefore $\int_0^2 (r \times s)\,dt = \int_0^2 \{-6t^3 i + 2(t^3 - 4t^2)j + 2t^4 k\}\,dt$

$$= -\left[\frac{3}{2}t^4\right]_0^2 i + \left[\frac{t^4}{2} - \frac{8t^3}{3}\right]_0^2 j + \left[\frac{2t^5}{5}\right]_0^2 k$$

$$= -24i - \frac{40}{3}j + \frac{64}{5}k$$

Example : If r(t) = t i − 3j + 2t k; s(t) = i − 2j + 2k and v(t) = 3i + t j − k; then find the value of

$$\int_1^2 r \cdot (s \times v)\,dt$$

Solution. s × v = (i − 2j + 2k) × (3i + t j − k)

$$= (2 - 2t)i + 7j + (t + 6)k$$

and   r · (s × v) = (t i − 3 j + 2t k) · {(2 − 2t)i + 7j + (t + 6)k}

$$= 14t - 21$$

Now $\int_1^2 r \cdot (s \times v)\,dt = \int_1^2 (14t - 21)\,dt$

$$= \left[7t^2 - 21t\right]_1^2 = -14 + 14 = 0$$

Example : If r(t) = 2i − j + 2k, when t = 2

$$= 4i - 2j + 3k \text{ when } t = 3$$

show that $\int_2^3 \left(r \cdot \dfrac{dr}{dt}\right) dt = 10$

Solution. From the standard formulae of integrals, we have

$$\int \left(r \cdot \frac{dr}{dt}\right) dt = \frac{1}{2}r^2 + c$$

$\therefore$ $\int_2^3 \left(r \cdot \dfrac{dr}{dt}\right) dt = \left[\dfrac{1}{2}r^2\right]_2^3$

$$= \frac{1}{2}\left[r^2 (\text{when } t = 3) - r^2 (\text{when } t = 2)\right] \qquad \text{...(1)}$$

when   t = 3 then r = 4i − 2j + 3k

and   r² = (4i − 2j + 3k)·(4i − 2j + 3k)

$$= 16 + 4 + 9 = 29 \qquad \text{...(2)}$$

Again when t = 2 then   r = 2i − j + 2k

and   r² = (2i − j + 2k)·(2i − j + 2k)

$$= 4 + 1 + 4 = 9 \qquad \text{...(3)}$$

Therefore from (1), $\int_2^3\left(r\cdot\dfrac{dr}{dt}\right)dt=\dfrac{1}{2}[29-9]$ 　　　　　　　　**[by (2) and (3)]**

$$=\dfrac{1}{2}(20)=10$$

Example : Integrate : $\dfrac{d^2r}{dt^2}=-n^2r$

Solution. Since r is not known in terms of t on RHS, therefore it can not be integrated.

Scalar multiplication on both sides by $2\dfrac{dr}{dt}$,

$$2\dfrac{dr}{dt}\cdot\dfrac{d^2r}{dt^2}=-2n^2r\,\dfrac{dr}{dt}$$

Now integrating both sides wrt t,

$$\int 2\dfrac{dr}{dt}\cdot\dfrac{d^2r}{dt^2}dt=-n^2\int 2r\cdot\dfrac{dr}{dt}dt$$

Using the earlier results. We have

$$\left(\dfrac{dr}{dt}\right)^2=-n^2(r)^2+c,$$

where c is a scalar constant.

Example : Find the value of r from the equation

$$\dfrac{d^2r}{dt^2}=a+t\,b$$

Given that : a and b are constant vectors and when t = 0, r = dr/dt = 0

Solution. Working method :

1. 　　　Solve the given equation treating vector variable as scalar variable.
2. 　　　Replace arbitrary scalars constants by arbitrary vector constants.

The given equation is

$$\dfrac{d^2r}{dt^2}=a+t\,b \hspace{4cm} ...(1)$$

Integrating both sides wrt t,

$$\dfrac{dr}{dt}=at+\dfrac{1}{2}bt^2+c \hspace{3.5cm} ...(2)$$

Again integrating wrt t,

$$r=\dfrac{1}{2}at^2+\dfrac{1}{6}bt^3+ct+d \hspace{3cm} ...(3)$$

Replacing scalar constants by vector constants in (2) and (3),

$$\dfrac{dr}{dt}=at+\dfrac{1}{2}bt^2+c \hspace{3.5cm} ...(4)$$

and 　　　$r=\dfrac{1}{2}at^2+\dfrac{1}{6}bt^3+ct+d \hspace{2.3cm} ...(5)$

By the given condition, when t = 0 then r = 0 and $\dfrac{dr}{dt}=0$

Therefore by (4),

    c = 0 and d = 0

Therefore the required solution is $r = \dfrac{1}{2}at^2 + \dfrac{1}{6}bt^3$

Example : The acceleration of a particle at any time $t \geq 0$ is given by :

$$a = \frac{dv}{dt} = 12 \cos 2t\ i - 8 \sin 2t\ j + 16t\ k$$

If the velocity v and displacement r are zero at t = 0; find v and r at any time.

Solution. By the given problem $\dfrac{dv}{dt} = 12 \cos 2t\ i - 8 \sin 2t\ j + 16t\ k$

Integrating, $v = i \int 12 \cos 2t\ dt - j \int 8 \sin 2t\ dt + k \int 16t\ dt$

or $v = 6 \sin 2t\ i + 4 \cos 2t\ j + 8t^2\ k + c$ ...(1)

Initial condition, when t = 0, then v = 0

Therefore $0 = 0i + 4j + 0k + c \qquad \Rightarrow \qquad c = -4\ j$

Hence from (1) the required velocity,

    $v = 6 \sin 2t\ i + (4 \cos 2t - 4)j + 8t^2\ k$ ...(2)

Again integration (1),

$$r = i \int 6 \sin 2t\ dt + 4j \int (\cos 2t - 1) dt + 8k \int t^2\ dt$$

$$= -3 \cos 2t\ i + 2(\sin 2t - 2t)j + \frac{8}{3}t^3\ k + d$$ ...(3)

Initial condition, when t = 0, then r = 0

Therefore $0 = -3i + 0j + 0k + d \Rightarrow d = 3i$

Therefore replacing from (3), the required

$$r = -3 \cos 2t\ i + 2(\sin 2t - 2t)j + \frac{8}{3}t^3 k + 3i$$

$$= 3(1 - \cos 2t)i + 2(\sin 2t - 2t)j + f\frac{8}{3}t^3\ k$$

6. LINE INTEGRAL

Definition : The integral obtained along any curve is called the line integral.

Let position vector of any point P on the curve C be r whose equation is r = f(t), where f(t) is the continuous and differential function.

The unit vector function along the tangent at any point P on the curve is $\dfrac{dr}{ds}$.

Let F(r) be a continuous vector point function. Now if the component of the vector function F(r) along the tangent at the point P be $F(r) \cdot \dfrac{dr}{ds}$ and the integral of $F(r) \cdot \dfrac{dr}{ds}$ i.e. $\displaystyle\int_c \left\{ F(r) \cdot \dfrac{dr}{ds} \right\} ds$ or $\displaystyle\int_c F(r) \cdot dr$

is called the line integral of the vector function F(r) on the curve C.

Cartesian form of Line Integral

Let $\qquad F(r) = F_1(t)i + F_2(t)j + F_3(t)k$

and $\qquad dr = dx \cdot i + dy \cdot j + dz \cdot k$, where $r = xi + yj + zk$

Therefore $\qquad \displaystyle\int_c F(r) \cdot dr = \int_c (F_1 i + F_2 j + F_3 k) \cdot (idx + jdy + kdz)$

or $\qquad \displaystyle\int_c F \cdot dr = \int_c (F_1 dx + F_2 dy + F_3 dz)$

Circulation

Definition : The integral of the vector F along any closed curve i.e. c is called the circulation of F around the closed curve C.

If the circulation of a vector point function in any region is equal to zero around every closed curve in that region, then that vector is called the ***Irrotational vector.***

Work done by a Force

Suppose a force F is a acting at a point and displaces it from a point A along the curve C. Let r be the position vector of the point A.

Since $\dfrac{dr}{ds}$ is the unit vector along the tangent at the point A in the direction s increasing, therefore

the component of F along the tangent at the point A = $F \cdot \dfrac{dr}{ds}$

Therefore the work done by F at C on the curve during the small displacement

$$\partial s = \left(F \cdot \frac{dr}{ds} \right) ds = F \cdot dr$$

Therefore total work done by $F = \displaystyle\int_c F \cdot dr$

Remark : If the path of integration is a closed curve, we write $\displaystyle\oint_c$ instead of $\displaystyle\int_c$

Example: Evaluate : $\displaystyle\int_c F \cdot dr$, where $F = (x^2 + y^2)i + xy\, j$ and C is the curve $y = x^2$ is xy-plane from (0, 0) to (3, 9).

Solution. For the curve C, the values of x varies from 0 to 3 and the values of y varies from 0 to 9. Therefore

$$\int_c F \cdot dr = \int_C \left[\left(x^2 + y^2 \right) i + xy \, j \right] \cdot \left(i \, dx + j \, dy \right) \qquad \{ \because \ r = xi + yj \}$$

$$= \int_C \left[\left(x^2 + y^2 \right) dx + xy \, dy \right]$$

$$= \int_C \left(x^2 + y^2 \right) dx + \int_C xy \, dy$$

$$= \int_{x=0}^{3} \left(x^2 + x^4 \right) dx + \int_{y=0}^{9} y^{3/2} dy \qquad \{ \because \ y = x^2 \}$$

$$= \left[\frac{1}{3} x^3 + \frac{1}{5} x^5 \right]_0^3 + \left[\frac{2}{5} y^{5/2} \right]_0^9$$

$$= \left[9 + \frac{243}{5} \right] + \left[\frac{2}{5} \cdot 243 \right]$$

$$= \frac{1}{5} \left[45 + 243 + 486 \right] = \frac{774}{5}$$

Example : Evaluate $\int_c F \cdot dr$, where $F = xy \, i + yz \, j + zx \, k$ and C is the curve $r = t \, i + t^2 \, j + t^3 k$, t varying form -1 to $+1$.

Solution. Around the curve C

$$r = xi + yj + zk \qquad \qquad \qquad \ldots(1)$$

Here $r = ti + t^2 j + t^3 k$ $\qquad \qquad \qquad \ldots(2)$

From (1) and (2),

$$x = t, \ y = t^2 \ \ \text{and} \ \ z = t^3 \qquad \qquad \ldots(3)$$

From (2), $\dfrac{dr}{dt} = i + 2t \, j + 3t^2 \, k$ $\qquad \qquad \ldots(4)$

Now $\int_c F \cdot dr = \int_c \left(F \cdot \dfrac{dr}{dt} \right) dt$

$$= \int_c \left\{ \left(xy \, i + yz \, j + zx \, k \right) \cdot \left(i + 2t \, j + 3t^2 \, k \right) \right\} dt \qquad \text{[by (4)]}$$

$$= \int_c \left\{ \left(t^3 \, i + t^5 \, j + t^4 \, k \right) \cdot \left(i + 2t \, j + 3t^2 \, k \right) \right\} dt \qquad \text{[by (3)]}$$

$$= \int_{-1}^{1} \left(t^3 + 2t^6 + 3t^6 \right) dt = \left[\frac{t^4}{4} + \frac{2}{7} t^7 + \frac{3}{7} t^7 \right]_{-1}^{1} = \frac{10}{7}$$

Example : Evaluate : $\int_c F \cdot dr$,

where $F = zi + xj + yk$, C is the arc of the curve $r = \cos t \, i + \sin t \, j + t \, k$ from $t = 0$ to $t = 2\pi$

Solution. From the equation of the curve,

$$dr = -\sin t \, i + \cos t \, j + k$$

The parametric equations of the curve

$$x = \cos t, \ y = \sin t, \ z = t \qquad \qquad \ldots(1)$$

$$\therefore \quad \int_C F \cdot dr = \int_C (z i + x j + y k) \cdot (-\sin t\, i + \cos t\, j + k)\, dt$$

$$= \int_C (-z \sin t + x \cos t + y)\, dt$$

$$= \int_0^{2\pi} \left(-t \sin t + \cos^2 t + \sin t\right) dt \qquad\qquad \text{[by (1)]}$$

$$= \left[(t \cos t - \sin t) + \left(\frac{1}{2} t + \frac{1}{4} \sin 2t\right) + (-\cos t) \right]_0^{2\pi}$$

$$= 2\pi + \pi - 1 + 1 = 3\pi$$

Example : Evaluate : $\int_C \vec{F} \cdot d\vec{r}$,

where $\vec{F} = C[-3a \sin\theta \cos\theta\, \hat{i} + a(2\sin\theta - 3\sin^2\theta)\hat{j} + b\sin 2\theta\, \hat{k}]$ and the curve C,

$\vec{r} = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j} + b\theta\,\hat{k}$,where θ varies from $\pi/4$ to $\pi/2$.

Solution. $\int_C \vec{F} \cdot d\vec{r} = \int_C F \cdot \dfrac{d\vec{r}}{d\theta}\, d\theta$

Given $\quad \vec{r} = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j} + b\theta\,\hat{k}$

$$\therefore \quad \frac{d\vec{r}}{d\theta} = -a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j} + b\hat{k}$$

$$\therefore \quad \vec{F} \cdot \frac{d\vec{r}}{d\theta} = C[3a^2 \sin^2\theta \cos\theta + a^2(2\sin\theta \cos\theta - 3\sin^2\theta \cos\theta) + b^2 \sin 2\theta]$$

$$= C[3a^2 \sin^2\theta \cos\theta + a^2\, 2\sin\theta \cos\theta - 3a^2 \sin^2\theta \cos\theta + b^2 \sin 2\theta]$$

$$= C(a^2 + b^2) \sin 2\theta$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\pi/4}^{\pi/2} C\left(a^2 + b^2\right) \sin 2\theta\, d\theta$$

$$= C\left(a^2 + b^2\right)\left[-\frac{\cos 2\theta}{2}\right]_{\pi/4}^{\pi/2} = C\left(a^2 + b^2\right)$$

Example : Evaluate : $\int_C F \cdot dr$, where $F = (x^2 + y^2)i - 2xy\, j$ and curve C is the rectangle in the xy plane bounded by x = 0, x = a, y = 0, y = b.

Solution. In xy plane,   z = 0,   therefore   r = xi + yj

$\therefore \quad$ dr = dx i + dy j

Now $\quad \int_C F \cdot dr = \int_C \left\{(x^2 + y^2)i - 2xy\, j\right\} \cdot (dx\, i + dy\, j)$

$$= \int_C \left\{(x^2 + y^2) dx - 2xy\, dy\right\}$$

According to the question, the path of integration C is the rectangle OABC formed by the given straight lines.

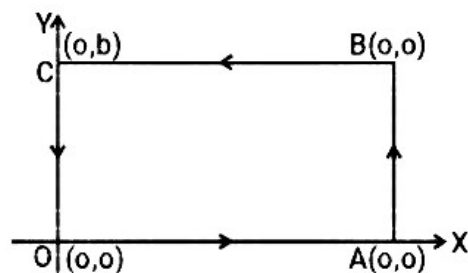**Fig.**

(a)　On the side OA of the rectangle $y = 0 \Rightarrow dy = 0$ and the limit of x is 0 to a.

Therefore along the side OA of the rectangle,

$$\int_c F \cdot dr = \int_0^a x^2 \, dx = \frac{1}{3} x^3 \qquad \qquad ...(1)$$

(b)　Similarly along the side AB of the rectangle

$$[x = a \Rightarrow dx = 0, \ y = 0, \ y = b]$$

$$\int_c F \cdot dr = \int_0^b (-2ay) \, dy = -2a \cdot \frac{1}{2} b^2 = -ab^2 \qquad \qquad ...(2)$$

(c)　along the side BC of the rectangle by $= b \Rightarrow dy = 0$, x = a to x = 0

$$\int_c F \cdot dr = \int_a^0 \left(x^2 + b^2 \right) dx = -\left(\frac{a^3}{3} + ab^2 \right) \qquad \qquad ...(3)$$

(d)　On the side CO of the rectangle $[x = 0 \Rightarrow dx = 0, \ y = b, \ y = 0]$

$$\int_c F \cdot dr = \int_c 0 \, dy = 0 \qquad \qquad ...(4)$$

Hence around the rectangle,

$$\int_c F \cdot dr = \frac{1}{3} a^3 - ab^2 - \left(\frac{a^3}{3} + ab^2 \right) + 0 = -2ab^2$$

Example : Evaluate : $\int_c \left(yz \, dx + (xz + 1) \, dy + xy \, dz \right)$, where C is any path from (1,0,0) to (2,1,4).

Solution. $\int_c \left\{ yz \, dx + (xz + 1) \, dy + xy \, dz \right\}$

$$= \int_c \left\{ yz \, i + (xz + 1) j + xy \, k \right\} \cdot (i \, dx + j \, dy + k \, dz)$$

$$= F \cdot dr,$$

where F = yz i + (xz + 1)j + xy k and r = xi + yj + zk

Let the given path be a straight line from to point (1, 0, 0) to the point (2, 1, 4), therefore

$$\frac{x-1}{2-1} = \frac{y-0}{1-0} = \frac{z-0}{4-0} = t \ \text{(say)}$$

then　x = t + 1, y = t, z = 4t

∴　　F = 4t² i + (4t² + 4t)j + (t² + t)k

and　　r = (t + 1)i + t j + 4t k

Therefore dr = (i + j + 4k)dt

∴　　From the point (1, 0, 0), t = 0 and for the point (2, 1, 4), t = 1

$\therefore \qquad \int_c F \cdot dr = \int_0^1 \left\{ 4t^2 i + \left(4t^2 + 4t \right) j + \left(t^2 + t \right) \right\} k \cdot \left\{ i + j + 4k \right\} dt$

$\qquad \qquad \qquad = \int_0^1 \left(4t^2 + 4t^2 + 4t + 4t^2 + 4t \right) dt$

$\qquad \qquad \qquad = \left[4t^3 + 4t^2 \right]_0^1 = 8.$

Example : Find the total work done in moving a particle in a force field given by F = 3xy i – 5zj + 10k along the curve x = t² + 1, y = 2t², z = t³ from t = 1 to 2.

Solution. We know that r = x i + y j + z k

Therefore dr = dx i + dy j + dz k

Now   F·dr = (3xy i – 5zj + 10k)·(dx i + dy j + dz k)

$\qquad \qquad$ = 3x y dx – 5z dy + 10x dz

From the equation of the curve,

$$\frac{dx}{dt} = 2t, \ \frac{dy}{dt} = 4t \ \text{and} \ \frac{dz}{dt} = 3t^2$$

Therefore $\int_c F \cdot dr = \int_1^2 \left(3xy \frac{dx}{dt} - 5z \frac{dy}{dt} + 10x \frac{dz}{dt} \right) dt$

$\qquad \qquad \qquad = \int_1^2 \left\{ 3 \left(t^2 + 1 \right) \left(2t^2 \right) (2t) - 5 \left(t^3 \right) (4t) + 10 \left(t^2 + 1 \right) \left(3t^2 \right) \right\} dt$

$\qquad \qquad \qquad = \int_1^2 \left(12t^5 + 10t^4 + 12t^3 + 30t^2 \right) dt = 303$

Example : If F = $\dfrac{y}{x^2 + y^2} i + \dfrac{x}{x^2 + y^2} j$; find the circulation of F in the anticlockwise direction round

the circle x² + y² = 1 situated in the z-plane.

Solution. The parametric equations of the curve

x = cos θ, y = sin θ, z = 0
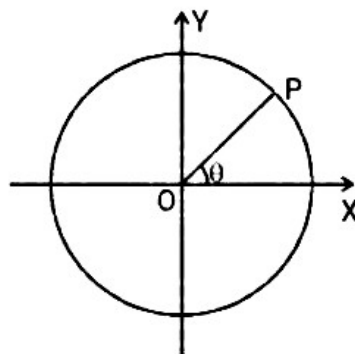
$\therefore \qquad$ r = cos θ i + sin θ j + 0k and $\dfrac{dr}{d\theta}$ = –sin θ i + cos θ j

Now by definition, circulation around the circle

$$\oint_c F \cdot dr = \oint_c F \cdot \frac{dr}{d\theta} d\theta$$

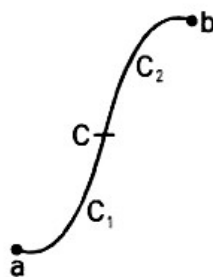$$= \oint_C \left\{ \frac{y}{x^2 + y^2} \sin\theta + \frac{x}{x^2 + y^2} \cos\theta \right\} d\theta$$

$$= \int_0^{2\pi} \left\{ \frac{-\sin^2\theta}{1} + \frac{\cos^2\theta}{1} \right\} d\theta = \int_0^{2\pi} (\cos 2\theta) d\theta = \left[\sin 2\theta \right]_0^{2\pi} = 0$$

Properties of the integral

(1) $\int_C k\, F \cdot dr = k \int_C F \cdot dr$, where k is a constant

(2) $\int_C (F + G) \cdot dr = \int_C F \cdot dr + \int_C G \cdot dr$

(3) $\int_C F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr$

where C is the sum of two curves C_1 and C_2.

(4) $\int_C F \cdot dr = -\int_{C'} F \cdot dr$, C' is the curve C taken in opposite sense.

or $\int_a^b F \cdot dr = -\int_b^a F \cdot dr$
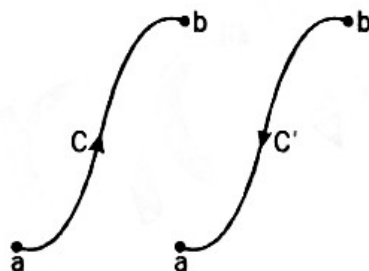
(5)   The line integral is independent of the choice of representation but depends on the path of integration.

$$\int_{C_1} F \cdot dr \neq \int_{C_2} F \cdot dr$$
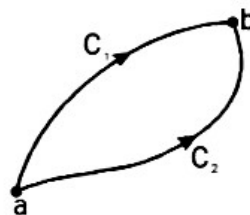
(6)   In case, the line integral depends only on the end points a and b, not on the path joining them, the vector field F is conservative vector field.

Recall that if $F = \nabla\phi$, F is conservative field and ϕ is its scalar potential

$$\int_a^b F \cdot dr - \int_a^b \nabla\phi \cdot dr = \int_a^b \left(\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k \right) \cdot (dx\, i + dy\, j + dz\, k)$$

$$= \int_a^b \left(\frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz \right)$$

$$= \int_a^b d\phi$$

$$\therefore \quad \int_a^b F \cdot dr = \phi(b) - \phi(a)$$

This formula should be applied whenever a line integral is independent of path.

Applications of Line Integral

(a) **Work done by a force :** The work done by a variable force F in the displacement along a curve C from point A to B is

$$\text{Work done} = \int_C F \cdot dr = \int_A^B F \cdot dr \qquad \text{...(1)}$$

Work done in conservative force field in moving a particle from A to B is independent of the path joining A and B and dependents only on the end points A and B.

In such case $F = \nabla\phi$, and hence

$$\text{Work done} = \int_A^B F \cdot dr = \int_A^B \nabla\phi \cdot dr = \phi(B) - \phi(A)$$

Also if A and B coincide, that is, C is any closed curve

$$\text{Work done} = \oint F.dr = 0 \qquad \text{...(2)}$$

(b) **Circulation due to a velocity field :** Let F denotes velocity of a fluid, then the circulation of F around a closed curve C is given by

$$\text{Circulation} = \oint_C F \cdot dr$$

(c) **Test for exact differential :** For $F = F_1 i + F_2 j + F_3 k$ the necessary and sufficient condition that $F_1 dx + F_2 dx + F_3 dz$ be an exact differential is that F must be conservative. i.e. $\nabla \times F = 0$.

$\Rightarrow$ These exist f such that $F = \nabla\phi$

Then $F_1 dx + F_2 dy + F_3 dz = F \cdot dr = \nabla\phi \cdot dr = d\phi = $ Exact differential $\qquad \text{...(3)}$

7. SURFACE AREA AND SURFACE INTEGRALS

A smooth surface S is called **Orientable** or **two sided** if it is possible to define a field n of unit normal vectors on S that varies continuously with position. Spheres and their smooth closed surface in space are **orientable**. By convention, we choose n on a closed surface to point outward.
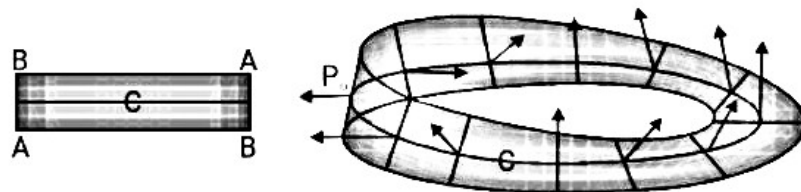

Fig. : Mobius strip (Not oriented surface)

The surface together with its normal field is called **oriented surface**. Unit normal is at any point of it is called the positive direction at the point.

Let f(x, y, z) = c be any oriented surface S and R be its projection (Shadow) on coordinate plane (say xy-plane)

The angle between S and R is given by

$$\cos\theta = \frac{\nabla f \cdot k}{|\nabla f||k|} = \frac{\nabla f \cdot k}{|\nabla f|} = \hat{n}.k \qquad \left[\because \hat{n} = \frac{\nabla f}{|\nabla f|} \text{ and } |k| = 1\right]$$

{z-axis is normal to xy-plane}

area of the surface f(x, y, z) = c over a closed and bounded plane region R is
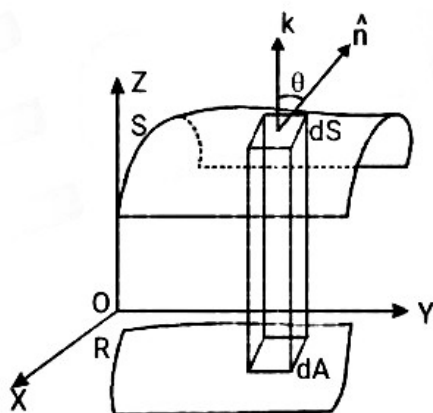
Fig.

$$dS \cos\theta = dA$$

$$\Rightarrow \quad dS = \frac{dA}{\cos\theta} = \frac{dA}{|\hat{n}.k|}$$

$$S = \iint_R \frac{|\nabla f|}{\nabla f \cdot k} dA$$

Surface Integral

If R be the projection (shadow) on xy-plane of surface S defined by equation f(x, y, z) = c and g is a continuous function defined at the points of S, then the integral of g over S is the integral.

$$\iint_S g(x,y,z) dS = \iint_R g(x,y,z)\frac{|\nabla f|}{\nabla f \cdot k} dA, \quad \nabla f \cdot k \neq 0$$

Suppose that F is a continuous vector field defined over an oriented surface S and that $\hat{n}$ is unit normal to the surface S. The surface integral of the normal component of F, i.e. $F \cdot \hat{n}$, is the flux of F across A in the positive direction and is given by

$$\text{Flux} = \iint_S F \cdot \hat{n} \, dS$$

$$= \iint_R (F \cdot \hat{n}) \frac{|\nabla f|}{(\nabla f \cdot k)} dA \qquad \left\{ \because dS = \frac{dA}{\cos \theta} \right\}$$

$$= \iint_R \left(F \cdot \frac{\nabla f}{|\nabla f|} \right) \frac{|\nabla f|}{(\nabla f \cdot k)} dA$$

$$= \iint_R \frac{F \cdot \nabla f}{\nabla f \cdot k} dA \qquad (\nabla f \cdot k \neq 0)$$

Remark :

(a)   To find the surface integral of a vector function or flux means the same thing.

(b)   If S is a surface defined by a function z = f(x, y) that has continuous first order partial derivatives throughout a region R in the xy-plane then

Let $\phi(x, y, z) = f(x, y) - z$ be the level surface

(a)   Unit normal to $\phi(x, y, z)$ is

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{f_x i + f_y j - k}{\sqrt{f_x^2 + f_y^2 + 1}}$$

(b)   Area of surface S is

$$S = \iint_R \frac{dA}{|\hat{n} \cdot k|}$$

$$= \iint_R \frac{\sqrt{f_x^2 + f_y^2 + 1}}{|-1|} dA$$

$$S = \iint_R \sqrt{f_x^2 + f_y^2 + 1} \, dx \, dy$$

Example : Evaluate $\iint_S A \cdot n \, ds$ where A = 18zi + 12j + 3yk and S is the part of the plane 2x + 3y + 6z = 12 which is located in the first octant.

Solution. We know that $ds = \dfrac{dA}{\cos \theta}$

where θ is angle between the normal to surface S and normal to its projection in xy-plane

$\therefore \quad \cos \theta = \hat{n} \cdot k$

$$= \frac{\nabla f}{|\nabla f|} \cdot k$$

S : f(x, y, z) = 2x + 3y + 6z − 12

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2i + 3j + 6k}{\sqrt{4 + 9 + 36}} = \frac{1}{7}(2i + 3j + 6k)$$

$\therefore \quad \cos\theta = \frac{1}{7}(2i + 3j + 6k) \cdot k = \frac{6}{7}$

Now $\quad \iint\limits_S A \cdot \hat{n}\, ds = \iint\limits_R \left\{ (18zi + 12j + 3yk) \cdot \frac{1}{7}(2i + 3j + 6k) \right\} \dfrac{dA}{6/7}$

$$= \int\limits_0^6 \int\limits_{y=0}^{\frac{12-2x}{3}} \frac{1}{6}(36z + 36 + 18y)\, dx\, dy$$

$$= \int\limits_0^6 \int\limits_0^{4-\frac{2}{3}x} (6z + 6 + 3y)\, dx\, dy$$

$$= \int\limits_0^6 \int\limits_0^{4-\frac{2}{3}x} (12 - 2x - 3y + 6 + 3y)\, dx\, dy$$

$$= \int\limits_0^6 \int\limits_0^{4-\frac{2}{3}x} (18 - 2x)\, dx\, dy$$

$$= \int\limits_0^6 \left[(18 - 2x)y \right]_0^{4-\frac{2}{3}x} dx$$

$$= \int\limits_0^6 (18 - 2x)\left(4 - \frac{2}{3}x \right) dx$$

$$= \left(72x - 20\frac{x^2}{2} + \frac{4}{3} \cdot \frac{x^3}{3} \right)_0^6$$

$= 432 - 360 + 96$

$= 72 + 96 = 168$
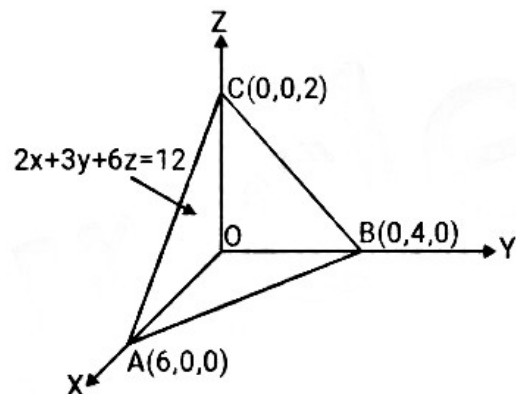
Example : Evaluate : $\iint\limits_S (\nabla \times F) \cdot n \, dS$ where $F = yi + (x - 2xz)j - xyk$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane.

Solution. $F = yi + (x - 2xz)j - xyk$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & x - 2xz & -xy \end{vmatrix}$$

$= i(-x + 2x) - j(-y - 0) + k(1 - 2z - 1)$

$= xi + yj - 2zk \qquad \qquad \qquad \qquad \dots(1)$
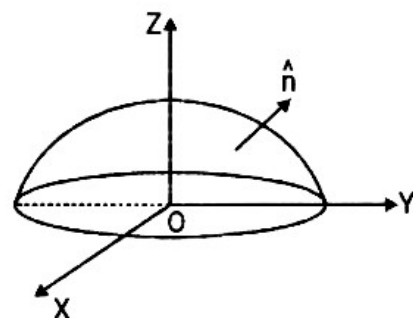
Let S be the surface given by $f(x, y, z) = x^2 + y^2 + z^2 - a^2$ above xy-plane.

Thus, the projection on xy-plane is the circle $x^2 + y^2 = a^2$. Also

$\hat{n}$ = unit normal to surface S

$$= \frac{\nabla f}{|\nabla f|} = \frac{2x\,i + 2y\,j + 2z\,k}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$= \frac{1}{a}(x\,i + y\,j + z\,k) \qquad \qquad ...(ii)$$

and $\quad dS = \dfrac{dx\,dy}{|\hat{n}\cdot k|} = \dfrac{dx\,dy}{z/a} \qquad \qquad ...(iii)$

Now $\quad \iint\limits_S (\nabla \times F)\cdot n\, dS = \iint\limits_R (x\,i + y\,j - 2z\,k)\cdot \dfrac{1}{a}(x\,i + y\,j + z\,k)\dfrac{dx\,dy}{z/a}$

$$= \iint\limits_R \frac{x^2 + y^2 - 2z^2}{z}\,dx\,dy$$

$$= \iint \frac{x^2 + y^2 - 2(a^2 - x^2 - y^2)}{\sqrt{a^2 - x^2 - y^2}}\,dx\,dy$$

$$= \int\limits_{-a}^{+a}\int\limits_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} \frac{3x^2 + 3y^2 - 2a^2}{\sqrt{a^2 - x^2 - y^2}}\,dx\,dy$$

$$= \int\limits_{-a}^{a}\int\limits_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} \frac{a^2 - 3(a^2 - x^2 - y^2)}{\sqrt{a^2 - x^2 - y^2}}\,dx\,dy$$

$$= \int\limits_{-a}^{a}\int\limits_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left\{ \frac{a^2}{\sqrt{a^2 - x^2 - y^2}} - 3\sqrt{a^2 - x^2 - y^2} \right\}dx\,dy$$

$$= \int\limits_{-a}^{a} \left(a^2 \sin^{-1}\frac{y}{\sqrt{a^2 - x^2}} - 3\frac{y}{2}\sqrt{a^2 - x^2 - y^2} - \frac{3(a^2 - x^2)}{2}\sin^{-1}\frac{y}{\sqrt{a^2 - x^2}} \right)_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= \int\limits_{-a}^{a} \left\{ a^2\frac{\pi}{2} - \frac{3}{2}(a^2 - x^2)\frac{\pi}{2} - a^2\frac{\pi}{2} - \frac{3}{2}(a^2 - x^2)\frac{\pi}{2} \right\}dx$$

$$= \frac{\pi}{2}\int\limits_{-a}^{a}(2a^2 - 3a^2 + 3x^2)\,dx$$

$$= \frac{\pi}{2}\left(x^3 - a^2 x \right)_{-a}^{a}$$

$$= \frac{\pi}{2}(a^3 - a^3 + a^3 - a^3) = 0$$

Example : Evaluate : $\iint_S A \cdot n \, dS$ over the entire surface S of the region bounded by the cylinder

$x^2 + z^2 = 9$, $x = 0$, $y = 0$, $z = 0$ and $y = 8$ where $A = 6zi + (2x + y)j - xk$.

Solution. Here the entire surface S consists of 5 surfaces, namely. S_1 : lateral surface of the cylinder ABCD, S_2 : AOED, S_3 : OBCE, S_4 : OAB, S_5 : CDE.

Thus, $\iint_S A \cdot \hat{n} \, dS = \iint_{S_1 + S_2 + S_3 + S_4 + S_5} A \cdot \hat{n} \, dS$

$$= \iint_{S_1} A \cdot n \, dS + \iint_{S_2} A \cdot n \, dS + ... + \iint_{S_5} A \cdot n \, dS$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 \text{ (say)}$$

S_1 : **ABCD** : The curved surface S_1 is $f = x^2 + z^2 = 9$. The unit outward normal to S_1 is

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2xi + 2zk}{\sqrt{4x^2 + 4z^2}} = \frac{xi + zk}{3}$$

$\therefore$ $A \cdot \hat{n} = \left[6zi + (2x + y)j - xk \right] k \cdot \dfrac{(xi + zj)}{3}$

$$= \frac{1}{3}(6xz - xz) = \frac{5}{3}xz$$

and $n \cdot \hat{k} = \dfrac{z}{3}$

$\therefore$ $\iint_{S_1} A \cdot \hat{n} \, dS = \int_{y=0}^{8} \int_{x=0}^{3} \frac{5}{3} xz \frac{dx \, dy}{z/3}$

$$= 5 \int_0^5 \int_0^3 x \, dx \, dy$$

$$= 5 \int_0^8 \int_0^3 x \, dx \, dy$$

$$= \frac{5 \times 9 \times 8}{2} = 180$$

Figure: Coordinate axes showing points E(0,8,0), C(0,8,3), D(3,8,0), B(0,0,3), A(3,0,0), O origin, and surface $x^2 + z^2 = 9$.
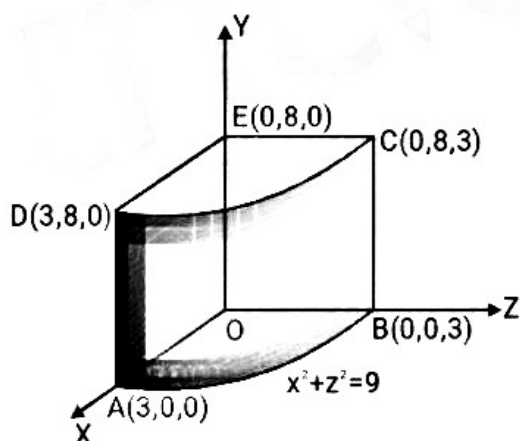
S_2 : **AOED** : The surface S_2 is xy-plane i.e. $z = 0$. Unit outward normal to the surface is

$$\hat{n} = -k.$$

$\therefore$ $\iint_{S_2} A \cdot n \, dS = \iint [6zi + (2x + y)j - xk] \cdot (-k) \dfrac{dx \, dy}{|-k \cdot k|}$

$$= \int_{y=0}^{8} \int_{x=0}^{3} x \, dx \, dy$$

$$= \left(\frac{x^2}{2} \right)_0^3 \cdot (y)_0^8 = \frac{9}{2} \times 8 = 36$$

S$_3$: OBCE : Surface S$_3$ is yz-plane i.e. x = 0. Unit outward normal to S$_3$ is $\hat{n} = -i$.

$$\therefore \quad \iint_{S_3} A \cdot \hat{n} \ dS = \int_{z=0}^{3} \int_{y=0}^{8} \left[6zi + (2x + y)j - xk \right] \cdot (-i) \frac{dy \ dz}{|-i \cdot i|}$$

$$= \int_{0}^{3} \int_{0}^{8} -6z \ dy \ dz$$

$$= -6 \left(\frac{z^2}{2} \right)_{0}^{3} (y)_{0}^{8} = -216$$

S$_4$: OAB : The section OAB is in xz-plane i.e. y = 0. The unit outward normal to S$_4$ is $\hat{n} = -j$.

$$\therefore \quad \iint_{S_4} A \cdot \hat{n} \ dS = \int_{0}^{3} \int_{0}^{\sqrt{9-x^2}} \left[6zi + (2x + y)j - xk \right] \cdot (-j) \frac{dx \ dz}{|-j \cdot j|}$$

$$= \int_{0}^{3} \int_{0}^{\sqrt{9-x^2}} -2x \ dx \ dz$$

$$= \int_{0}^{3} -2x (z)_{0}^{\sqrt{9-x^2}} \ dx$$

$$= \int_{0}^{3} -2x\sqrt{9 - x^2} \ dx$$

$$= \left[\frac{\left(9 - x^2 \right)^{3/2}}{\frac{3}{2}} \right]_{0}^{3}$$

$$= \frac{2}{3}(-27) = -18$$

S$_5$: CDE : The section S$_5$ is parallel to xz-plane, y = 8. The unit outward normal to S$_5$ is $\hat{n} = j$.

$$\iint_{S_5} A \cdot \hat{n} \ dS = \int_{0}^{3} \int_{0}^{\sqrt{9-x^2}} \left[6zi + (2x + y)j - xk \right] \cdot (j) \frac{dx \ dz}{|j \cdot j|}$$

$$= \int_{0}^{3} \int_{0}^{\sqrt{9-x^2}} (2x + 8) dx \ dz$$

$$= \int_{0}^{3} (2x + 8)(z)_{0}^{\sqrt{9-x^2}} \ dx$$

$$= 2 \int_{0}^{3} (x + 4)\sqrt{9 - x^2} \ dx$$

$$= 2\left[\frac{(9-x^2)^{3/2}}{3/2\cdot(-2)} + 4\cdot\frac{x}{2}\sqrt{9-x^2} + 4\cdot\frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3$$

$$= 2\left(\frac{27}{3} + 18\cdot\frac{\pi}{2}\right)$$

$$= 18(1 + \pi)$$

Thus, the required surface integral is

$$\iint_S A\cdot n\ dS = 180 + 36 - 216 - 18 + 18 + 18\pi = 18\pi$$

Example : Find the flux of the vector field

A = (x − 2z)i + (x + 3y + z)j + (5x + y)k through the upper side of the triangle ABC with vertices at the points A(1, 0, 0), B(0, 1, 0) and C(0, 0, 1).

Solution. Equation of the plane containing the given triangle ABC is

$$f(x, y, z) \equiv x + y + z - 1$$

Unit normal n̂ to ABC is

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{i+j+k}{\sqrt{1+1+1}}$$

$$= \frac{1}{\sqrt{3}}(i+j+k)$$
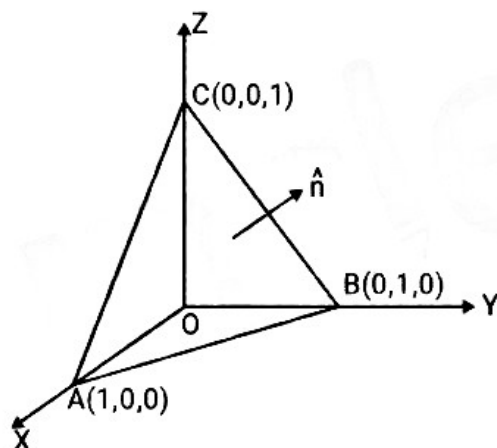
Flux of A $= \iint_S A\cdot\hat{n}\ dS = \iint_S A\cdot\hat{n}\ \frac{dA}{|\hat{n}\cdot k|}$

$$= \iint_{AOB}\left[(x-2z)i+(x+3y+z)j+(5x+y)k\right]\cdot\frac{(i+j+k)}{\sqrt{3}}\cdot\frac{dx\ dy}{\frac{1}{\sqrt{3}}}$$

$$= \int_0^1\int_0^{1-x}(x-2z)+(x+3y+z+5x+y)dx\ dy$$

$$= \int_0^1\int_0^{1-x}\left[7x+4y-(1-x-y)\right]dx\ dy$$

$$= \int_0^1\left[(8x-1)y+\frac{5y^2}{2}\right]_0^{1-x}dx$$

$$= \int_0^1\left[(8x-1)(1-x)+\frac{5}{2}(1-x)^2\right]dx$$

$$= \left[\frac{-11}{2}\frac{x^3}{3}+2x^2+\frac{3}{2}x\right]_0^1 = \frac{-11}{6}+2+\frac{3}{2} = \frac{5}{2}$$

Example : Find the surface area of the plane 2z + x + 2y = 12 cut off by x = 0, y = 0, x = 1, y = 1.

Solution. Let $\phi(x, y, z) = \dfrac{12 - x - 2y}{2} - z = 0$ be the level surface.

$$\frac{\partial \phi}{\partial x} = \phi_x = \frac{-1}{2}, \ \frac{\partial \phi}{\partial y} = \phi_y = -1$$

$\therefore$

$$S = \iint_R \sqrt{\phi_x^2 + \phi_y^2 + 1}\, dx\, dy$$

$$= \int_0^1 \int_0^1 \sqrt{\frac{1}{4} + 1 + 1}\, dx\, dy$$

$$= \frac{3}{2} \int_0^1 (x)_0^1\, dy$$

$$= \frac{3}{2} \int_0^1 dy$$

$$= \frac{3}{2} (y)_0^1$$

$$= \frac{3}{2}$$
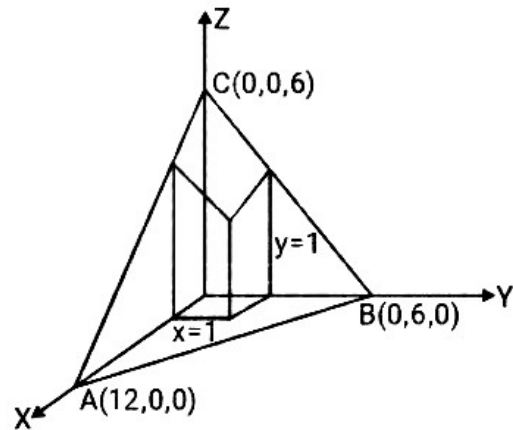
8. VOLUME INTEGRAL

Let V be a region is space enclosed by a closed surface S. Let F be a vector point function, and ϕ be a scalar point function, then the triple integrals

$$\iiint_V F \, dV \quad \text{and} \quad \iiint_V \phi \, dV$$

are known as *volume integral* or *space integrals*.

In component form

$$\iiint_V F \, dV = i \iiint_V F_1 \, dx \, dy \, dz + j \iiint_V F_2 \, dx \, dy \, dz + k \iiint_V F_3 \, dx \, dy \, dz$$

Example : If V is the region of first octant bounded by $y^2 + z^2 = 9$ and the plane $x = 2$ and $F = 2x^2 yi - y^2 j + 4xz^2 k$. Then evaluate $\iiint_V (\nabla \cdot F) dV$

Solution. $\nabla \cdot F = \left(\dfrac{\partial}{\partial x} i + \dfrac{\partial}{\partial y} j + \dfrac{\partial}{\partial z} k \right) \cdot \left(2x^2 yi - y^2 j + 4xz^2 k \right)$

$= 4xy - 2y + 8xz$
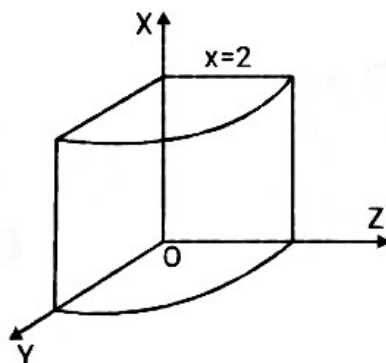
$$\iiint_V (\nabla \cdot F) dV = \iiint_V (4xy - 2y + 8xz) dx \, dy \, dz$$

$$= \int_{x=0}^{2} \int_{y=0}^{3} \int_{z=0}^{\sqrt{9-y^2}} (4xy - 2y + 8xz) dx \, dy \, dz$$

$$= \int_{0}^{2} \int_{0}^{3} \left(4xy \, z - 2y \, z + 4x \, z^2 \right)_{0}^{\sqrt{9-y^2}} dx \, dy$$

$$= \int_{0}^{2} \int_{0}^{3} \left\{ (4x - 2) y \sqrt{9 - y^2} + 4x (9 - y^2) \right\} dx \, dy$$

$$= \int_{0}^{2} \left\{ (4x - 2) \left[\frac{-1}{3} (9 - y^2)^{3/2} \right]_{0}^{3} + 4x \left(9y - \frac{y^3}{3} \right) \right\}_{0}^{3} dx$$

$$= \int_{0}^{2} \left\{ (4x - 2) 9 + 4x (27 - 9) \right\} dx$$

$$= \int_0^2 (108x - 18)\,dx$$

$$= \left(\frac{108x^2}{2} - 18x\right)_0^2 = 216 - 36 = 180.$$

Example : If F = 2z i + xj + yk. Evaluate $\iiint_V F\,dV$ where V is the region bounded by the surfaces x = 0, y = 0, x = 2, y = 4, z = x^2, z = 2.

Solution. $\iiint_V F\,dV = \int_{y=0}^{4}\int_{x=0}^{2}\int_{z=x^2}^{2}(2z\,i - x\,j + y\,k)\,dz\,dx\,dy$

$$= \int_0^4\int_0^2 \left(z^2 i - xz\,j + yz\,k\right)_{x^2}^{2}\,dx\,dy$$

$$= \int_0^4\int_0^2 \left\{4i - 2x\,j + 2y\,k - \left(x^4 i - x^3 j + x^2 y\,k\right)\right\}\,dx\,dy$$

$$= \int_0^4 \left[\left(4x - \frac{x^5}{5}\right)i + \left(\frac{x^4}{4} - x^2\right)j + \left(2xy - \frac{x^3 y}{3}\right)k\right]_0^2\,dy$$

$$= \int_0^4 \left\{\left(8 - \frac{32}{5}\right)i + (4 - 4)j + \left(4y - \frac{8}{3}y\right)k\right\}\,dy$$

$$= \int_0^4 \left(\frac{8}{5}y\,i + \frac{2}{3}y^2 k\right)_0^4$$

$$= \frac{32}{5}i + \frac{32}{5}k$$

$$= \frac{32}{15}(3i + 5k)$$

Example : Evaluate $\iiint_V \nabla \times A\,dV$ where A = (x + 2y)i – 3z j + xk and V is the closed region in the first octant bounded by the plane 2x + 2y + z = 4

Solution. $\nabla \times A = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+2y & -3z & x \end{vmatrix} = i(0+3) - j(1-0) + k(0-2)$

$$= 3i - j - 2k$$

$$\iiint_V \nabla \times A\,dV = \iiint (3i - j - 2k)\,dx\,dy\,dz$$

$$= \int\limits_{x=0}^{2} \int\limits_{y=0}^{2-x} \int\limits_{z=0}^{4-2y-2x} (3i - j - 2k) \, dx \, dy \, dz$$

$$= \int\limits_{0}^{2} \int\limits_{0}^{2-x} (3i - j - 2k)(z)_{0}^{4-2y-2k} \, dx \, dy$$

$$= (3i - j - 2k) \int\limits_{0}^{2} \int\limits_{0}^{2-x} 2(2 - y - x) \, dx \, dy$$

$$= 2(3i - j - 2k) \int\limits_{0}^{2} \left\{ (2-x)y - \frac{y^2}{2} \right\}_{0}^{2-x} dx$$

$$= 2(3i - j - 2k) \int\limits_{0}^{2} \frac{(2-x)^2}{2} \, dx$$

$$= (3i - j - 2k) \left(\frac{(2-x)^3}{-3} \right)_{0}^{2}$$

$$= \frac{8}{3}(3i - j - 2k)$$
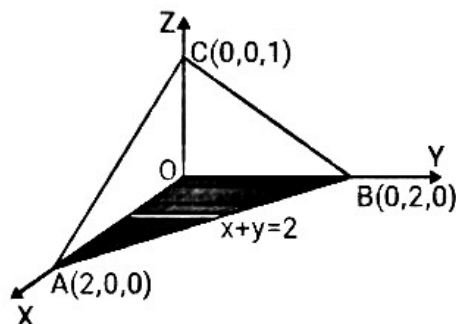
Example : Find the volume enclosed between the two surfaces $S_1 : z = 8 - x^2 - y^2$ and $S_2 : z = x^2 + 3y^2$.

Solution. Eliminating z from the two surfaces S_1 and S_2, we get

$$8 - x^2 - y^2 = x^2 + 3y^2$$

$$\Rightarrow \quad x^2 + 2y^2 = 4$$

Thus, the two surface intersect on the elliptic cylinder.

So the solid region between S_1 and S_2 is covered when

z varies from $x^2 + 3y^2$ to $8 - x^2 - y^2$.

y varies from $-\sqrt{\dfrac{4-x^2}{2}}$ to $\sqrt{\dfrac{4-x^2}{2}}$,

x varies from –2 to 2.

$$\therefore \quad \iiint\limits_{V} dV = \int\limits_{-2}^{2} \int\limits_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int\limits_{x^2+3y^2}^{8-x^2-y^2} dx \, dy \, dz$$

$$= \int\limits_{-2}^{2} \int\limits_{-\sqrt{\frac{4-x^2}{2}}}^{+\sqrt{\frac{4-x^2}{2}}} (z)_{x^2+3y^2}^{8-x^2-y^2} \, dx \, dy$$

$$= \int\limits_{-2}^{2} \int\limits_{-\sqrt{\frac{4-x^2}{2}}}^{+\sqrt{\frac{4-x^2}{2}}} (8 - x^2 - y^2 - x^2 - 3y^2) \, dx \, dy$$

$$= \int_{-2}^{2} \left(8y - 2x^2 y - \frac{4y^3}{3} \right) \Bigg|_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \int_{-2}^{2} \left[2(8 - 2x^2)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right] dx$$

$$= \int_{-2}^{2} \left\{ 4\frac{\left(4-x^2\right)^{3/2}}{\sqrt{2}} - \frac{4}{3}\frac{\left(4-x^2\right)^{3/2}}{\sqrt{2}} \right\} dx$$

$$= \frac{16}{3\sqrt{2}} \int_{0}^{2} \left(4-x^2\right)^{3/2} dx$$

$$= \frac{8\sqrt{2}}{3} \cdot \int_{0}^{1} \cdot 8(1-t)^{3/2} \cdot \frac{2}{2\sqrt{t}} dt \quad \text{Put } x^2 = 4t \ 2x \ dx = 4t$$

$$= \frac{64\sqrt{2}}{3} \frac{\frac{1}{2}\frac{5}{2}}{\frac{6}{2}} = \frac{64\sqrt{2}}{3 \cdot 2} \cdot \pi \cdot \frac{3}{2}\frac{1}{2}\sqrt{\pi} = 8\pi\sqrt{2}$$

9. GAUSS'S DIVERGENCE THEOREM

Statement : The normal surface integral of a function F over the boundary of a closed region is equal to the volume integral of div F taking throughout the region.

If F is a continuously differentiable vector point function in a region V enclosed by the closed surface S, then :

$$\int_S F \cdot \hat{n} \, dS = \int_V (\nabla \cdot F) \, dV \qquad \qquad ...(1)$$

where $\hat{n}$ is the unit outward drawn normal vector to the surface S.

The **Cartesian form** of (1) is as follows :

$$\iint_S (F_1 dy \, dz + F_2 dz \, dx + F_3 dx \, dy) = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx \, dy \, dz, \qquad ...(2)$$

where F_1, F_2 and F_3 are scalar components of F along the axes.

Remark : The volume integral is reduced into surface integral with the help of this theorem.

Conversely, the surface integral can also be reduced into volume integral.

Proof. Let V be the region wrt the axes such that if any straight line is drawn parallel to any axis, then that will meet the surface S in two points only.

In fig., A_3 is the projection of region V on the plane XOY. Let the coordinates of any point R in the plane A_3 be (x, y, 0).

Let any line through R meets the surfaces at the points P and Q.

Therefore z-co-ordinate of Q by $\psi(x, y)$ and that of P be $\phi(x, y)$.

Since RP > RQ, therefore $\phi(x, y) > \psi(x, y)$

Fig.
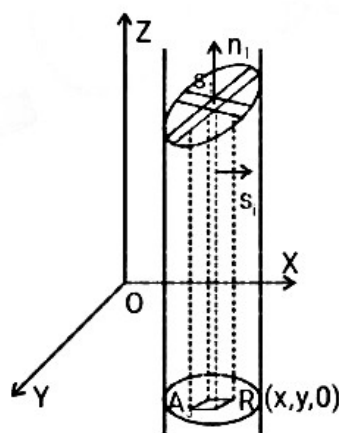
Now $\iiint_V \frac{\partial F_3}{\partial z} dx \, dy \, dz = \iint_{A_3} \left[\int_\psi^\phi \frac{\partial F_3}{\partial z} dz \right] dx \, dy$

$= \iint_{A_3} \left[F_3(x, y, \phi) - F_3(x, y, \psi) \right] dx \, dy$

$= \iint_{A_3} F_3(x, y, \phi) dx \, dy - \iint_{A_3} F_3(x, y, \psi) dx \, dy \qquad ...(3)$

Let S_1 and S_2 be the sub-parts of the surface S.

Let $\hat{n}$ be the outwards drawn unit normal vector at any point on the surface S.

Therefore dx dy = dS cos θ = n̂·k dS,

where θ is the angle of the normal with x-axis.

$$\therefore \quad \iint_{A_3} F_3(x, y, \phi)\, dx\, dy = \int_{S_1} F_3\, \hat{n} \cdot k\, dS \qquad \ldots(4)$$

$$\text{and} \quad \iint_{A_3} F_3(x, y, \psi)\, dx\, dy = -\int_{S_1} F_2\, \hat{n} \cdot k\, dS \qquad \ldots(5)$$

The negative sign of equation (5) shows that the normal drawn outwards make an obtuse angle with z-axis.

The positive sign in equation (4) is because the outwards normal on the surface S_1 make an acute angle with z-axis.

Substituting the value from the equations (4) and (5) in (3)

$$\iiint_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz = \int_{S_1} F_3\, \hat{n} \cdot k\, dS + \int_{S_2} F_3\, \hat{n} \cdot k\, dS$$

$$= \int_S F_3\, \hat{n} \cdot k\, dS \qquad \ldots(6)$$

Similarly,

$$\iiint_V \frac{\partial F_2}{\partial y}\, dx\, dy\, dz = \int_S F_2\, \hat{n} \cdot j\, dS \qquad \ldots(7)$$

$$\text{and} \quad \iiint_V \frac{\partial F_1}{\partial x}\, dx\, dy\, dz = \int_S F_1\, \hat{n} \cdot i\, dS \qquad \ldots(8)$$

Adding the equations (6), (7) and (8),

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\, dy\, dz$$

$$= \int_S \left(F_1 i + F_2 j + F_3 k \right) \cdot \hat{n}\, dS$$

$$\int_V (\nabla \cdot F)\, dV = \int_S F \cdot \hat{n}\, dS$$

This theorem is also true for the region V enclosed by two closed curves S_1 and S_2, where the surface S_2 is situated inside the surface S_1.

Remark : The volume integral is reduced into surface integral with the help of this theorem. Conversely, the surface integral can also be reduced into volume integral.

Fig.
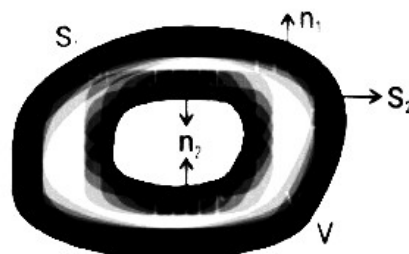
Example : Evaluate :

$$\iint_S F \cdot \hat{n}\, dS, \text{ where } F = 4xz\, i - y^2\, j + yz\, k$$

S is the surface of the cube bounded by the planes :

Solution. By Gauss's theorem,

$$\iint_S \mathbf{F} \cdot \hat{n}\, ds = \iiint \nabla \cdot \mathbf{F}\, dV$$

$$= \iiint \left\{ \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz) \right\} dV$$

$$= \iiint (4z - y)\, dV$$

$$= \int_0^1 \int_0^1 \int_0^1 (4z - y)\, dx\, dy\, dz \qquad\qquad [\because \ dV = dx\, dy\, dz]$$

$$= \int_0^1 \int_0^1 \left(2z^2 - yz\right)_0^1 dx\, dy = \int_0^1 \int_0^1 (2 - y)\, dx\, dy$$

$$= \int_0^1 \left[2y - \frac{y^2}{2} \right]_0^1 dx = \int_0^1 \frac{3}{2}\, dx = \left[\frac{3}{2} x \right]_0^1 = \frac{3}{2}$$

Example : Evaluate :

$$\int_S \left(y^2 z^2 \mathbf{i} + z^2 x^2 \mathbf{j} + z^2 y^2 \mathbf{k} \right) \cdot \hat{n}\, dS,$$

where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy plane.

Solution. $\mathbf{F} = y^2 z^2 \mathbf{i} + z^2 x^2 \mathbf{j} + z^2 y^2 \mathbf{k}$

$$\therefore \qquad \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\left(y^2 z^2\right) + \frac{\partial}{\partial y}\left(z^2 x^2\right) + \frac{\partial}{\partial z}\left(z^2 y^2\right)$$

$$= 2zy^2$$

By Gauss's theorem,

Fig.
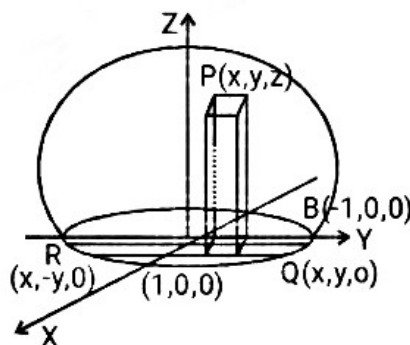
$$\iint_S \mathbf{F} \cdot \hat{n}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV$$

$$= \iiint_V \left(2zy^2\right) dx\, dy\, dz,$$

This is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy plane i.e., hemisphere above the xy plane and the limits for x, y, z are as follows :

First integrate wrt z from z = 0 to $z = \sqrt{\left(1 - x^2 - y^2\right)}$, then integrate wrt y, between the limit $y = -\sqrt{\left(1 - x^2\right)}$ to $y = \sqrt{\left(1 - x^2\right)}$ and finally wrt x between the limits x = −1 to x = 1.

$\therefore$ Required integral $= 2 \iiint_V zy^2 \, dx \, dy \, dz$

$$= \int_{x=-1}^{1} \int_{y=-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} \left[y^2 z^2 \right]_0^{\sqrt{(1-x^2-y^2)}} \, dx \, dy$$

$$= \int_{-1}^{1} \int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} \left(1 - x^2 - y^2 \right) y^2 \, dx \, dy$$

$$= \int_{-1}^{1} \left[\left(1 - x^2 \right) \frac{y^3}{3} - \frac{y^5}{5} \right]_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} \, dx$$

$$= 2 \int_{-1}^{1} \left[\frac{\left(1 - x^2 \right) \left(1 - x^2 \right)^{3/2}}{3} - \frac{\left(1 - x^2 \right)^{5/2}}{5} \right] \, dx$$

$$= \frac{4}{15} \int_{-1}^{1} \left(1 - x^2 \right)^{5/2} \, dx$$

put $x = \sin \theta \Rightarrow dx = \cos \theta \, d\theta$

$$= \frac{4}{15} \int_{-\pi/2}^{\pi/2} \cos^5 \theta \, \cos \theta \, d\theta$$

$$= \frac{8}{15} \int_{0}^{\pi/2} \cos^6 \theta \, d\theta$$

$$= \frac{8}{15} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{12}$$

Example : Use Gauss's divergence theorem to show that :

$$\iint_S \left(x \, dy \, dz + y \, dz \, dx + z \, dx \, dy \right) = 4\pi a^2 ,$$

where the surface S is the sphere $x^2 + y^2 + z^2 = a^2$.

Solution. We know that

$$\iint_S \left(F_1 \, dy \, dz + F_2 \, dz \, dx + F_3 \, dx \, dy \right)$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx \, dy \, dz ,$$

where $F_1 = x$, $F_2 = y$, and $F_3 = z$

$\therefore \quad \dfrac{\partial F_1}{\partial x} = 1, \dfrac{\partial F_2}{\partial y} = 1$ and $\dfrac{\partial F_3}{\partial z} = 1$

$\therefore$ From equation (1)

$$\iint_S \left(x \, dy \, dz + y \, dz \, dx + z \, dx \, dy \right)$$

$$= \iiint_V (1 + 1 + 1) dx \, dy \, dz , \text{ enclosed by surface S, where V is volume.}$$

$$= 3 \iiint_V dx \, dy \, dz = 3 \text{ (volume of the sphere)}$$

$$= 3 \cdot (4/3)\pi a^2 = 4\pi a^3$$

Example : If V is the volume enclosed by any closed surface S; show that:

(a) $\int_S \hat{n}\, dS = 0$ (b) $\int_S (r \times \hat{n})\, dS = 0$

(c) $\int_S \hat{n} \cdot (\nabla \times F)\, dS = 0$ (d) $\int_S r \cdot \hat{n}\, dS = 3V$

Solution. (a) If F be any constant vector, then

$$F \cdot \int_S \hat{n}\, dS = \int_S F \cdot \hat{n}\, dS \qquad\qquad [\because \ \text{F is a constant vector}]$$

By Gauss's theorem, $\int_S F \cdot \hat{n}\, dS = \int_V \nabla \cdot F\, dV$

But $\quad \nabla \cdot F = \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z} \right) \cdot F = 0$

$$F \cdot \int_S \hat{n}\, dS = \int_V \nabla \cdot F\, dV = 0$$

or, $\quad \int_S \hat{n}\, dS = 0 \qquad\qquad [\because \ \text{F is a constant vector}]$

(b) If F be any constant vector, then

$$F \cdot \int_S r \times \hat{n}\, dS = \int_S F \cdot (r \times \hat{n})\, dS$$

$$= \int (F \times r) \cdot \hat{n}\, dS \qquad\qquad \text{(by interchanging the scalar and vector products)}$$

$$= \int_V \nabla \cdot (F \times r)\, dS \qquad\qquad [\text{by Gauss's theorem}] \qquad\qquad ...(1)$$

But $\quad \nabla \cdot (F \times r) = r \cdot (\nabla \times F) - F \cdot (\nabla \times r) = 0$

$$[\because \ \nabla \times F = 0, \ \because \ \text{F is constant and } \nabla \times r = 0]$$

By (1), $\quad F \cdot \int_S (r \times \hat{n})\, dS = 0$

or, $\quad \int_S r \times \hat{n}\, dS = 0$

(c) By Gauss's theorem,

$$\int_S \hat{n} \cdot (\nabla \times F)\, dS = \int_V \nabla \cdot (\nabla \times F)\, dV$$

$$= 0 \qquad\qquad [\because \ \nabla \cdot (\nabla \times F) = 0]$$

(d) By Gauss's theorem,

$$\iint_S r \cdot \hat{n}\, dS = \iiint_V (\nabla \cdot r)\, dV = \iiint_V 3\, dV \qquad\qquad (\because \ \nabla \cdot r = 3)$$

$$= 3V$$

Example : Evaluate :

$$\int_S F \cdot \hat{n}\, dS, \quad \text{where} \quad F = zi + xj + 3y^2z\, k,$$

where S is the surface of the cylinder $x^2 + y^2 = 16$ include in the first octant between z = 0 and z = 5.

Solution. The projection of the given surface on xz plane will be a rectangle whose sides will be x, z axes and lines parallel to these.

Therefore the given integral

$$\iint_S F \cdot \hat{n} \, dS = \iint_{Rectangle} F \cdot \hat{n} \frac{dx \, dz}{|\hat{n} \cdot j|} \qquad \qquad ...(1)$$

Normal vector on the given surface $x^2 + y^2 = 16$

$$\nabla \phi = \left(i\frac{\partial}{x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right)(x^2 + y^2) = 2xi + 2yj$$

$$\therefore \qquad \hat{n} = \frac{2(xi + yj)}{\sqrt{(4x^2 + 4y^2)}} = \frac{2(xi + yj)}{2\sqrt{16}} = \frac{1}{4}(xi + yj)$$

$$\therefore \qquad \hat{n} \cdot j = \frac{1}{4}y \text{ and } F \cdot \hat{n} = (zi + xj - 3y^2zk) \cdot \frac{1}{4}(xi + yj)$$

$$= \frac{1}{4}(xz + xy)$$

Now by (1),

$$\int_C F \cdot \hat{n} \, dS = \frac{1}{4} \iint x(z + y) \frac{dx \, dz}{\frac{1}{4}y} = \int_{z=0}^{5} \int_{x=0}^{4} \left(x + \frac{xz}{y} \right) dx \, dz$$

$$= \int_0^5 \int_0^4 \left\{ x + xz(16 - x^2)^{-1/2} \right\} dx \, dz$$

$$= \int_{x=0}^{4} \left\{ xz + \frac{x}{\sqrt{(16 - x^2)}} \cdot \frac{z^2}{2} \right\}_0^5 dx$$

$$= \int_0^4 \left\{ 5x + \frac{25}{2} \frac{x}{\sqrt{(16 - x^2)}} \right\} dx = \left[\frac{5x^2}{2} - \frac{25}{2}\sqrt{(16 - x^2)} \right]_0^4$$

$$= 5 \times 8 - \frac{25}{2}(0 - 4) = 90$$

Example : If V is the volume enclosed by any closed surface S; show that :

$$\iint_S F \cdot \hat{n} dS = 6V, \quad \text{where} \quad F = xi + 2yj + 3zk$$

Solution. Given F = xi + 2yj + 3zk

$$\therefore \qquad \nabla \times F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot (xi + 2yj + 3zk)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z)$$

$$= 1 + 2 + 3 = 6$$

∴ By Gauss's theorem,

$$\int_S F \cdot \hat{n} \, dS = \int_V (\nabla \cdot F) dV = \int_V 6 \, dV = 6V$$

Example : Verify Gauss's Divergence theorem and show that :

$$\int_S F \cdot \hat{n} \, ds = \frac{1}{3} a^5, \quad \text{where} \quad F = (x^3 - yz)i - 2x^2yj + 2k$$

S is the surface of the cube bounded by the co-ordinate planes :

$$x = y = z = 0; \quad x = y = z = a$$

Solution. Here $\nabla \cdot F = \dfrac{\partial}{\partial x}(x^3 - yz) + \dfrac{\partial}{\partial y}(-2x^2y) + \dfrac{\partial}{\partial z}(2)$

$$= 3x^2 - 2x^2 + 0 = x^2$$

$$\therefore \quad \int_S F \cdot \hat{n} \, dS = \int_V \nabla \cdot F \, dV = \int_0^a \int_0^a \int_0^a x^2 \, dx \, dy \, dz$$

$$= \int_0^a \int_0^a x^2 \, dx \, dy \, [z]_0^a$$

$$= a\int_0^a x^2 \, dx \, [y]_0^a = a^2 \left[\frac{x^3}{3}\right]_0^a = \frac{1}{3}a^5$$

Verification by direct integration : In figure outward unit normal have been shown on each face of the cube. Now on the face ABCD

$$\hat{n} = i, \; x = a, \; dS = dy \, dz$$

$$\int_S F \cdot \hat{n} \, dS = \iint_S \left\{ (x^3 - yz)i - 2x^2yj + 2k \right\} \cdot i \, dS$$

$$= \int_0^a \int_0^a (x^3 - yz) dy \, dz = \int_0^a \int_0^a (a^3 - yz) dy \, dz \qquad (\because \; x = a)$$

$$= a^3 \int_0^a y \, dy \, [z]_0^a - \int_0^a y \, dy \left[\frac{z^2}{2}\right]_0^2$$

$$= a^3 \left[\frac{1}{2}y^2\right]_0^a - \frac{1}{2}a^2 \left[\frac{1}{2}y^2\right]_0^a$$

$$= a^5 - \frac{1}{4}a^4$$

Taking $\hat{n} = -i$ on the face OEFC,

$$\int_S F \cdot \hat{n} \, dS = \iint_S \left\{ (x^3 - yz)i - 2x^2yj + 2k \right\} \cdot (-i) dS$$

$$= -\int_0^a \int_0^a yz \, dy \, dz$$

$\therefore$ On this face $x = 0$

$$= \left[\frac{y^2}{2}\right]_0^a \left[\frac{z^2}{2}\right]_0^a = \frac{1}{4}a^4$$

Taking $\hat{n} = j$ on the face BEFC.

$$\int F \cdot \hat{n} \, dS = \int_S \left\{ \left(x^3 - yz \right) i - 2x^2 y j + 2k \right\} \cdot j \, dS$$

$$= \int_0^a \int_0^a \left(-2x^2 y \right) dx \, dz$$

$$= -2 \int_0^a \int_0^a x^2 a \, dx \, dz \qquad \left[\because \quad \text{on this face } y = a \right]$$

$$= -2a \int_0^a x^2 \, dx \left[z \right]_0^a = -\left(2/3 \right) a^5$$
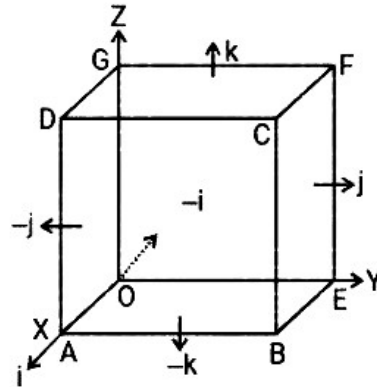
Fig.

Taking $\hat{n} = k$ on the face CDGF,

$$\int_S F \cdot \hat{n} \, dS = \iint_S \left\{ \left(x^3 - yz \right) i - 2x^2 y j + 2k \right\} \cdot k \, dS$$

$$= \int_0^a \int_0^a 2ax \, dy = 2 \int_0^a dx \left[y \right]_0^a$$

$$= 2a \left[x \right]_0^a = 2a^2$$

Taking $\hat{n} = -k$ on the face ABEO,

$$\int_S F \cdot \hat{n} \, dS = \iint_S \left\{ \left(x^3 - yz \right) i - 2x^2 y j + 2k \right\} \cdot \left(-k \right) dS$$

$$= \int_0^a \int_0^a -2dx \, dy = -2 \int_0^a dx \left[y \right]_0^a = -2a \left[x \right]_0^a$$

$$= -2x^2$$

Adding the integrals on all the faces,

$$= a^5 - (1/4)a^4 + (1/4)a^4 - (2/3)a^5 + 0 + 2a^2 - 2a^2$$

$$= \frac{1}{3} a^5$$

Example : Evaluate :

$$\int_S F \cdot \hat{n} \, dS, \quad \text{where} \quad F = xi + yj + z^2 k.$$

where S is the closed surface bounded by the cone $x^2 + y^2 = z^2$ and the plane $z = 1$.

Solution. Here $\nabla \cdot F = \dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial y}(y) + \dfrac{\partial}{\partial z}\left(z^2 \right)$

$= 1 + 1 + 2z = 2(1 + z)$

$\int_s F \cdot \hat{n} \, dS = \int_V \nabla \cdot F \, dV$

$= \int_V 2(1+z) \, dV$

$= 2\int_V dV + 2\int_V z \, dV$

$= 2V + 2V\bar{z},$ $\left\{ \because z = \dfrac{\int z \, dV}{V} \right\}$
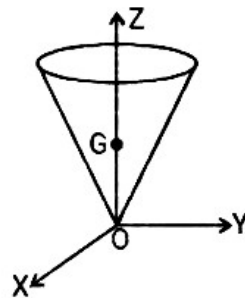
Fig.

where V is the volume of the given cone and $\bar{z}$ is the centre of gravity of the cone (situated on the axis of the one) which is at a distance y from the vertex O. The base of the given cone is a circle whose radius is 1 and height is also 1.

∴ $OG = 3/4, \quad V = (1/3)\pi 1^2 \cdot 1 = \pi/3$

∴   Required integral $= 2V(1+\bar{z})$

$$= 2\frac{\pi}{3}\left(1+\frac{3}{4}\right) = \frac{2\pi}{3} \cdot \frac{7}{4} = \frac{7\pi}{6}$$

10. STOKE'S THEOREM

Statement : The line integral of a vector function F around any closed curve is equal to the surface integral of curl F taken over any surface of which of the curve is a boundary edge.

If F be any continuous differentiable vector function and S is the surface enclosed by a curve C, then :

$$\int_C F \cdot dr = \iint_S \hat{n} \cdot (\nabla \times F) dS,$$

where $\hat{n}$ is the unit normal vector at any point of S and drawn in the sense in which a right handed screw would move rotated in the sense of description of C.

Cartesian form :

$$\int_C F \cdot dr = \iint_S (i\,F_1 + j\,F_2 + k\,F_3) \cdot (i\,dx + j\,dy + k\,dz) = \int_C F_1 dx + F_2 dy \qquad [\because \ dz = 0]$$

Since $\hat{n}$ = k, therefore

$$n = \int_S (\nabla \times F) \cdot \hat{n}\ dS = \int (\nabla \times F) \cdot k\ dS = \iint_S \left(\frac{\partial F_1}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\ dy$$

Proof. (a) For surfaces in the planes :

Let the region S_1 be subdivided into sub-regions S_i such that if a straight line is drawn parallel to any axis then that will meet the curve C_i atmost in two points.

Let C_i be situated between the lines x = a and x = b.

Let any line the be drawn parallel to y-axis meets the curve at the point P and Q, where the ordinate of P is y = $\phi(x)$ and that of Q is y = $\psi(x)$. In the figure limit of the curve has been divided by the line PQ in two parts C_1 and C_2

Now
$$\iint_{S_i} \frac{\partial F_1}{\partial y} dx\ dy = \int_a^b \left\{ \int_{y=\phi(x)}^{y=\psi(x)} \frac{\partial F_1}{\partial y} dy \right\} dx$$

$$= \int_a^b \left\{ F_1(x,y) \right\}_{y=\phi(x)}^{y=\psi(x)}$$

$$= \int_a^b \left[F_1\{x,\psi(x)\} - F_1\{x,\phi(x)\} \right] dx$$

$$= \int_a^b F_1\{x,\psi(x)\}\, dx - \int_a^b F_1\{x,\phi(x)\}\, dx$$

$$= -\int_b^a F_1\{x,\psi(x)\}\, dx - \int_b^a F_1\{x,\phi(x)\}\, dx$$

$$= -\int_{C_2} F_1(x,y)\, dx - \int_{C_1} F_1(x,y)\, dx$$

$$= -\int_{C_i} F_1(x,y)\, dx \qquad \qquad \qquad ...(1)$$

Fig.

Similarly, $\iint_{S_i} \dfrac{\partial F_2}{\partial x} dx\, dy = \int_{C_i} F_2(x,y)\, dy$ ...(2)
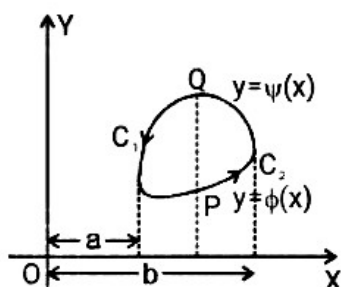
From (1) and (2),

$$\int_{C_i} (F_1 dx + F_2 dy) = \iint_{S_i}\left(\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}\right) dx\, dy$$

or, $\int_C F \cdot dr = \iint_S (\nabla \times F)\cdot \hat{n}\, dS$

(b) The Cartesian form of the surface in **space**,

$$\iint_S (\nabla \times F)\cdot \hat{n}\, dS = \iint_S \left(\dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z}\right) dy\, dz + \iint_S \left(\dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_2}{\partial x}\right) dz\, dx + \iint_S \left(\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}\right) dx\, dy$$

$$= \iint_D \left(\dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z}\right)\dfrac{\partial(y,z)}{\partial(u,v)} + \iint_D \left(\dfrac{\partial F_3}{\partial z} - \dfrac{\partial F_1}{\partial x}\right)\dfrac{\partial(z,x)}{\partial(u,v)}$$

$$+ \iint_D \left(\dfrac{\partial F_1}{\partial x} - \dfrac{\partial F_2}{\partial y}\right)\dfrac{\partial(x,y)}{\partial(u,v)}, \qquad \text{[*by Jacobian]} \qquad ...(1)$$

where D is any region in u-v plane whose image is S.

Taking the terms of F_1 in the equation (1),

$$\dfrac{\partial F_1}{\partial z}\cdot\dfrac{\partial(z,x)}{\partial(u,v)} - \dfrac{\partial F_1}{\partial y}\cdot\dfrac{\partial(x,y)}{\partial(u,v)} = -\dfrac{\partial F_1}{\partial z}\dfrac{\partial(x,z)}{\partial(u,v)} - \dfrac{\partial F_1}{\partial y}\dfrac{\partial(x,y)}{\partial(u,v)} - \dfrac{\partial F_1}{\partial x}\dfrac{\partial(y,x)}{\partial(u,v)}$$

(where the additional last term = 0)

$$= -\dfrac{\partial F_1}{\partial z}\left\{\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial z}{\partial v} - \dfrac{\partial x}{\partial v}\cdot\dfrac{\partial z}{\partial u}\right\} - \dfrac{\partial F_1}{\partial y}\left\{\dfrac{\partial x}{\partial u}\cdot\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\cdot\dfrac{\partial y}{\partial u}\right\} - \dfrac{\partial F_1}{\partial x}\left\{\dfrac{\partial x}{\partial u}\cdot\dfrac{\partial x}{\partial v} - \dfrac{\partial x}{\partial v}\cdot\dfrac{\partial x}{\partial u}\right\}$$

$$= -\dfrac{\partial x}{\partial u}\left\{\dfrac{\partial F_1}{\partial z}\cdot\dfrac{\partial z}{\partial v} + \dfrac{\partial F_1}{\partial y}\cdot\dfrac{\partial y}{\partial u} + \dfrac{\partial F_1}{\partial z}\cdot\dfrac{\partial x}{\partial v}\right\} + \dfrac{\partial x}{\partial v}\left\{\dfrac{\partial F_1}{\partial x}\cdot\dfrac{\partial x}{\partial u} + \dfrac{\partial F_1}{\partial y}\cdot\dfrac{\partial y}{\partial u} + \dfrac{\partial F_1}{\partial z}\cdot\dfrac{\partial z}{\partial u}\right\}$$

Similarly, the values of the terms F_2 and F_3 in (1) are respectively

$$\dfrac{\partial F_2}{\partial u}\cdot\dfrac{\partial y}{\partial v} - \dfrac{\partial F_2}{\partial v}\cdot\dfrac{\partial y}{\partial u} \text{ and } \dfrac{\partial F_3}{\partial u}\cdot\dfrac{\partial z}{\partial v} - \dfrac{\partial F_3}{\partial v}\cdot\dfrac{\partial z}{\partial u}$$

Therefore the RHS of (1)

$$\iint_D \left\{\left(\dfrac{\partial F_1}{\partial u}\cdot\dfrac{\partial x}{\partial v} - \dfrac{\partial F_1}{\partial v}\cdot\dfrac{\partial x_1}{\partial u}\right) + \left(\dfrac{\partial F_2}{\partial u}\cdot\dfrac{\partial y}{\partial v} - \dfrac{\partial F_2}{\partial v}\cdot\dfrac{\partial y}{\partial u}\right) + \left(\dfrac{\partial F_3}{\partial u}\cdot\dfrac{\partial z}{\partial v} - \dfrac{\partial F_3}{\partial v}\cdot\dfrac{\partial z}{\partial u}\right)\right\} du\, dv$$

Now by Stoke's theorem,

$$\iint_D \left\{\left(\dfrac{\partial F_1}{\partial v}\cdot\dfrac{\partial x}{\partial v} - \dfrac{\partial F_1}{\partial v}\cdot\dfrac{\partial x}{\partial u}\right) du\, dv\right.$$

$$= \int_{C_i}\left(F_1 \dfrac{\partial x}{\partial u}\cdot du + F_1 \dfrac{\partial x}{\partial v}\cdot dv\right) = \int_{C_i} F_1\, dx$$

Similarly,

$$\iint_D \left\{ \left(\frac{\partial F_2}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial F_2}{\partial v} \cdot \frac{\partial y}{\partial u} \right) du \ dv = \int_{C_1} F_2 \ dy \right.$$

and $$\iint_D \left\{ \left(\frac{\partial F_3}{\partial u} \cdot \frac{\partial z}{\partial v} - \frac{\partial F_3}{\partial v} \cdot \frac{\partial z}{\partial u} \right) du \ dv = \int_{C_1} F_3 \ dz \right.$$

Therefore $\int_{C_1} (\nabla \times F) \cdot \hat{n} \ dS = \int_{C_1} (F_1 dx + F_2 dy + F_3 dz) = \int_{C_1} F \cdot dr$

$$\int_S (\nabla \times F) \cdot \hat{n} \ dS = \int_C F \cdot dr$$

Now applying Stoke's theorem for each subregion and adding the **results,**

$$\int_S (\nabla \times F) \cdot \hat{n} \ dS = \int_C F \cdot dr$$

Example : Verify Stoke's theorem for the function F = zi + xj + yk, where the curve C is the unit circle in the xy-plane bounding the hemisphere z = $\sqrt{(1 - x^2 - y^2)}$.

Solution. Given F = zi + xj – yk   and   r = xi + yj + zk

∴   F·dr = (zi + xj + yk)·(i dx + j dy + k dz)

or,   F·dr = z dx + x dy + y dz   ...(1)

The unit circle C in xy plane $x^2 + y^2 = 1$, z = 0

∴   For the curve C, z = 0 and dz = 0

∴   by (1), F·dr = x dy   ...(2)

From the fig., x = cos ϕ, y = sin ϕ, where ϕ varies from 0 to 2π.
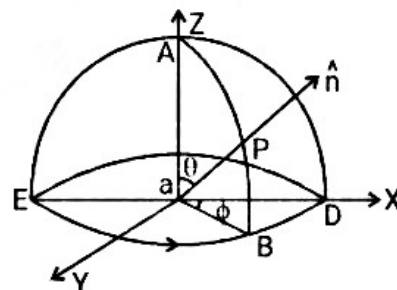
∴   From (2),

Fig.

$$\int_C F \cdot dr = \int_C x \ dy = \int_0^{2\pi} \cos \phi (\cos \phi) d\phi$$

$$= \frac{1}{2} \int_0^{2\pi} 2 \cos^2 \phi \ d\phi$$

$$= \frac{1}{2} \int_0^{2\pi} (1 + \cos 2\phi) d\phi$$

$$= \frac{1}{2} \left[\phi + \frac{1}{2} \sin 2\phi \right]$$

$$= \frac{1}{2} [2\pi] = \pi \qquad \qquad ...(3)$$

From the fig, the direction cosines of the line OP are

$\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta$

Therefore $\hat{n}$ = $(\sin \theta \cos \phi)i + (\sin \theta \sin \phi)j + (\cos \theta)k$,

where $\hat{n}$ is outer normal at the point P.

Now $\quad \nabla \times F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$

$= i\left[\dfrac{\partial}{\partial y}(y) - \dfrac{\partial}{\partial z}(x)\right] + j\left[\dfrac{\partial}{\partial z}(z) - \dfrac{\partial}{\partial x}(y)\right] + k\left[\dfrac{\partial}{\partial x}(x) - \dfrac{\partial}{\partial y}(z)\right]$

$= i + j + k$

$\therefore \quad \hat{n} \cdot (\nabla \times F) = [(\sin \theta \cos \phi)i + (\sin \theta \sin \phi)j + (\cos \theta)k] \cdot [i + j + k]$

$= \sin \theta \cos \phi + \sin \theta \sin \phi + \cos \theta$

$\therefore \quad \int_S \hat{n} \cdot (\nabla \times F)dS = \int_S (\sin \theta \cos \phi + \sin \theta \sin \phi + \cos \theta)dS,$ where dS $= \sin \theta \ d\phi \ d\theta$

$= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} (\sin \theta \cos \phi + \sin \theta \sin \phi + \cos \theta)\sin \theta \ d\theta \ d\phi$

$= \int_0^{\pi/2} \int_0^{2\pi} \sin^2 \theta (\cos \phi + \sin \phi)d\theta \ d\phi + \int_0^{\pi/2} \int_0^{2\pi} \sin \theta \cos \theta \ d\theta \ d\phi$

$= \dfrac{1}{2} \int_0^{\pi/2}(1 - \cos 2\theta)[\sin \phi \cos \phi]_0^{2\pi} \ d\theta + \dfrac{1}{2}\int_0^{\pi/2} \sin 2\theta [\phi]_0^{2\pi} \ d\theta$

$= \dfrac{1}{2}\left[\theta - \dfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} \times (0) + \dfrac{1}{2}\left[-\dfrac{1}{2}\cos 2\theta\right]_0^{\pi/2}(2\pi)$

$= \dfrac{1}{2}(\pi/2)(0) + \dfrac{1}{2}\left(\dfrac{1}{2} + \dfrac{1}{2}\right)(2\pi) = 0 + \pi = \pi$ ...(4)

Therefore from (3) and (4),

$\int_C F \cdot dr = \int_S \hat{n} \cdot (\nabla \times F)dS$

Example : Using Stoke's theorem, evaluate :

$\int_C xy \ dx + xy^2 \ dy,$

where C is the square in the xy plane with vertices respectively :

(1, 0), (−1, 0), (0, 1); (0, 1)

Solution. Writing $\int_C (xy \ dx + xy^2 dy)$ as $\int_C (xyi + xy^2 j) \cdot (i \ dx + j \ dy)$,

$\int_C (xy i + xy^2 j)dr,$ where   dr = xi + yj

Here   F = xy i + xy² j

$$\therefore \quad \nabla \times F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix} = \left(y^2 - x\right)k$$

Taking $\hat{n} = k$ and $dS = dx\ dy$,

$$\hat{n} \cdot (\nabla \times F) = k \cdot \left(y^2 - x\right)k = y^2 - x$$

By Stokes's theorem,

$$\int_c \left(xy\,i + xy^2\,j\right) \cdot dr = \iint_s \hat{n} \cdot (\nabla \times F)\,dS$$

$$= \iint_s \left(y^2 - x\right)dx\ dy$$

$$= \int_{x=-1}^{1} \int_{y=-1}^{1} \left(y^2 - x\right)dx\ dy = \int_{-1}^{1} \left[\frac{1}{3}y^3 - xy\right]_{-1}^{1} dx$$

$$= \int_{-1}^{1} \left[\left(\frac{1}{3} - x\right) - \left(-\frac{1}{3} + x\right)\right]dx$$

$$= \int_{-1}^{1} \left(\frac{2}{3} - 2x\right)dx = \left[\frac{2}{3}x - x^2\right]_{-1}^{1} = \frac{4}{3}$$

Example : Verify Stoke's theorem for the function $F = x^2 i + xy\ j$ integrated round the square in the plane $z = 0$, whose sides are along the lines $x = y = 0$ and $x = y = 0$.

Solution. In the plane $z = 0$, $r = xi + yj$

Therefore

$$dr = i\ dx + j\ dy$$

$$\therefore \quad F \cdot dr = (x^3 i + xy\ j) \cdot (i\ dx + j\ dy)$$

$$= x^2\ dx + xy\ dy$$

Now $\quad \int_c F \cdot dr = \int_c \left(x^2 dx + xy\ dy\right)$
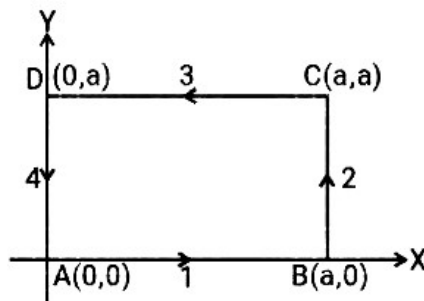
Fig.

(a) On the line AB, $y = 0$, Therefore $dy = 0$

$$\int_c F \cdot dr = \int_0^a x^2\ dx = \frac{1}{3}a^3 \qquad \qquad \qquad ...(1)$$

(b) On the line BC, x = a, Therefore dx = 0

∴   Along the side BC of the square $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^a ay \, dy = \frac{1}{2}a^3$ ...(2)

(c) On the line CD, y = a, Therefore dy = 0

∴   Along the side CD of the square $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^a x^2 \, dx = -\frac{1}{3}a^3$ ...(3)

(d) On the line DA, x = 0, Therefore dx = 0

∴   Along the side DA of the square $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ ...(4)

Therefore from (1), (2), (3) and (4), along the square ABCD

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{3}a^3 + \frac{1}{2}a^3 - \frac{1}{a}a^3 + 0 = \frac{1}{2}a^3 \qquad \text{...(5)}$$

Check : curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix} = y\mathbf{k}$

On square ABCD, $\hat{n} = \mathbf{k}$, Therefore curl $\mathbf{F} \cdot \hat{n} = y\mathbf{k} \cdot \mathbf{k} = y$

∴ $\int_s$ curl $\mathbf{F} \cdot \hat{n} \, dS = \int_0^a \int_0^a y \, dx \, dy$

$$= \int_0^a dx \left[\frac{1}{2}y^2 \right]_0^a = \frac{1}{2}a^2 \left[x \right]_0^a = \frac{1}{2}a^3 \qquad \text{...(6)}$$

From (5) and (6), $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_s$ curl $\mathbf{F} \cdot \hat{n} \, dS$

Hence the Stoke's theorem is verified.

Example : Evaluate by Stoke's theorem

$$\int_C \left(e^x dx + 2y \, dy - dz \right),$$

where C is the curve $x^2 + y^2 = 4$ and z = 2.

Solution. Writing $\int_C \left(e^x dx + 2y \, dy - dz \right)$

or $\int_C \left(e^x \mathbf{i} + 2y \mathbf{j} - \mathbf{k} \right) \cdot \left(\mathbf{i} dx + \mathbf{j} dy + \mathbf{k} \, dz \right),$

$\int_C \left(e^x \mathbf{i} + 2y \mathbf{j} - \mathbf{k} \right) \cdot d\mathbf{r}$, where $d\mathbf{r} = \mathbf{i} \, dx + \mathbf{j} \, dy + \mathbf{k} \, dz$

Therefore taking $\mathbf{F} = e^x \mathbf{i} + 2y \mathbf{j} - \mathbf{k}$ and by Stoke's theorem

$$\int_C \left(e^x \mathbf{i} + 2y \mathbf{j} - \mathbf{k} \right) \cdot d\mathbf{r} = \int_s \hat{n} \cdot (\nabla \times \mathbf{F}) dS, \qquad \text{...(1)}$$

where the surface S is the boundary C of the curves, $x^2 + y^2 = 4$ and z = 2.

Here C is a circle $x^2 + y^2 = 4$ and z = 2 whose centre is D(0, 0, 2) and radius is 2 and C is also the boundary of S_1, therefore $\hat{n} = \mathbf{k}$

∴ $\hat{n} \cdot \nabla \times \mathbf{F} = \mathbf{k} \cdot \nabla \times \left(e^x \mathbf{i} + 2y \mathbf{j} - \mathbf{k} \right)$ ...(2)

Now $\nabla \times F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$

$= i(0) + j(0) + k(0) = 0$

From (2), $\hat{n} \cdot (\nabla \times F) = 0$

Therefore from (1), the given integral

$$= \int_{S_1} \hat{n} \cdot (\nabla \times F) dS = 0$$
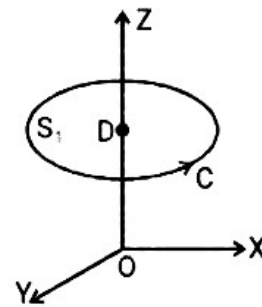
Fig.

Example : Prove by Stoke's theorem :

(a) $\displaystyle \int_c \phi(\nabla\psi) \cdot dr = -\int_c \psi(\nabla\phi) \cdot dr$

(b) $\displaystyle \int_c \phi(\nabla\psi) dr = \int_S (\nabla\phi \times \nabla\psi) \cdot \hat{n}\, dS$

Solution. (a) By Stoke's theorem

$$\int_c F \cdot dr = \int_c (\nabla \times F) \cdot \hat{n}\, dS \qquad \ldots(1)$$

Let $F = \nabla(\phi\psi)$, therefore $\nabla \times F = \nabla \times (\nabla\phi\psi)$

or, $\nabla \times F = 0$ $[\because \ \nabla \times (\nabla\phi\psi) = 0]$

Therefore from (1), $\displaystyle \int_c \nabla(\phi\psi) \cdot dr = 0 \qquad \ldots(2)$

and $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

From (2), $\displaystyle \int_c (\phi\nabla\psi + \psi\nabla\phi) \cdot dr = 0$

or, $\displaystyle \int_c \phi(\nabla\psi) \cdot dr = -\int_c \psi(\nabla\phi) \cdot dr$

(b) $\displaystyle \int_c F \cdot dr = \int_S (\nabla \times F) \cdot \hat{n}\, dS$ (By Stoke's theorem) $\ldots(1)$

Let $F = \phi\nabla\psi$

$\therefore \quad \nabla \times F = \nabla \times (\phi\nabla\psi)$

$= (\nabla\phi) \times \nabla\psi + \phi\nabla \times (\nabla\psi)$ $[\because \ \nabla \times fV = (\nabla\phi) \times V + \phi(\nabla \times V)]$

or, $\nabla \times F = \nabla\phi \times \nabla\psi$ $[\because \ \nabla \times (\nabla\psi) = 0] \qquad \ldots(2)$

Therefore from (1) and (2)

$$\int_c \phi\nabla\psi \cdot dr = \int_S (\nabla\phi \times \nabla\psi) \cdot \hat{n}\, dS$$

11. GREEN'S THEOREM

Statement : If ϕ and ψ are two continuously differentiable vector point functions such that $\nabla\phi$ and $\nabla\psi$ are also continuously differentiable within V enclosed by a surface S, then

$$\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) dV = \int_S \left(\phi\nabla\psi - \psi\nabla\phi\right)\cdot\hat{n}\, dS$$

Proof. Using Gauss theorem for the vector point function $\phi\nabla\psi$

$$\int_S \phi\nabla\psi\cdot\hat{n}\, dS = \int_C \nabla\cdot(\phi\nabla\psi)dV \qquad\qquad ...(1)$$

But $\qquad \nabla\cdot(\phi\nabla\psi) = \phi\nabla\cdot(\nabla\psi) + \nabla\phi\cdot\nabla\psi \qquad \{\because \quad \nabla\cdot(\phi A) = \phi(\nabla\cdot A) + \nabla\phi\cdot A\}$

$$= \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi \qquad\qquad ...(2)$$

From (1) and (2),

$$\int_S \phi\nabla\psi\cdot\hat{n}\, dS = \int_V \nabla\phi\cdot\nabla\psi\, dV + \int_V \psi\nabla^2\phi\, dV \qquad\qquad ...(3)$$

Interchanging ϕ and ψ,

$$\int_V \psi\nabla\phi\cdot\hat{n}\, dS = \int_V \nabla\psi\cdot\nabla\phi\, dV + \int_V \psi\nabla^2\phi\, dV \qquad\qquad ...(4)$$

Subtracting (4) and (3),

$$\int_S \left(\phi\nabla\psi - \psi\nabla\phi\right)\cdot\hat{n}\, dS = \int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)dV \qquad\qquad ...(5)$$

which is the Green's theorem

Another Form :

Since $\quad \int_S \left(\phi\nabla\psi - \psi\nabla\phi\right)\cdot\hat{n}\, dS = \int_S \left(\phi\dfrac{\partial\psi}{\partial\eta} - \psi\dfrac{\partial\phi}{\partial\eta}\right)dS,$

where $\dfrac{\partial\psi}{\partial\eta}$ is the derivative of ψ in the direction of outer normal at the point ψ.

Therefore, $\nabla\psi = \dfrac{\partial\psi}{\partial\eta}\cdot\hat{n}, \quad \nabla\phi = \dfrac{\partial\phi}{\partial\eta}\cdot\hat{n}$

From (5), $\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)dV = \int_S \left(\phi\dfrac{\partial\psi}{\partial\eta} - \psi\dfrac{\partial\phi}{\partial\eta}\right)dS$

Cartesian form of Green's Theorem :

If C is a regular closed curve in xy plane enclosing a region S, P(x, y) and Q(x, y) be two continuously differentiable functions in the region S, then :

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\, dy = \int_C (P\, dx + Q\, dy)$$

Solution. Let F = Pi + Qj, Since $\hat{n}$ = k, therefore

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$$

$$= -i\frac{\partial Q}{\partial z} + j\frac{\partial P}{\partial z} + k\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$\therefore \quad \nabla \times F \cdot \hat{n} = \nabla \times F \cdot k = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$\therefore \quad \int_S \nabla \times F \cdot \hat{n}\ ds = \int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\ dy \qquad \qquad ...(1)$$

and $\quad \int_c F \cdot dr = \int (Pi + Qi) \cdot (i\,dx + j\,dy + k\,dz)$

$$= \int_c (P\ dx + Q\ dy) \qquad \qquad ...(2)$$

From (1) and (2),

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\ dy = \int_c (P\ dx + Q\ dy) \qquad \qquad ...(3)$$

Cor. : When Q = x and P = –y, then $\dfrac{\partial P}{\partial y} = -1, \dfrac{\partial Q}{\partial x} = 1$

Therefore from (3), $\int_c (-y\ dx - x\ dy) = \iint_A (1+1) dx\ dy$

(If A is the area of the enclosed region by the curve C)

$$= \iint_A 2\ dx\ dy = 2A$$

$$\therefore \quad A = \frac{1}{2}\int_c (x\ dy - y\ dx)$$

Example : Evaluate by Green's theorem :

$$\int_c \left(e^{-x} \sin y\ dx + e^{-x} \cos y\ dy\right)$$

where C is the rectangle with vertices (π, 0), (0, 0), (π, π/2) and (0, π/2)

Solution. Here P = $e^{-x} \sin y$; Q = $e^{-x} \cos y$

$$\therefore \quad \frac{\partial P}{\partial y} = e^{-x} \cos y \qquad \text{and} \qquad \frac{\partial Q}{\partial x} = -e^{-x} \cos y$$

$$\therefore \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = e^{-x}\left[-\cos y - \cos y\right] = -2e^{-x} \cos y$$

Therefore by Green's theorem,

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\ dy = -2\int_{x=0}^{\pi}\int_{y=0}^{\pi/2} e^{-x} \cos y\ dx\ dy$$

$$= -2\int_0^{\pi} e^{-x}\ dx\left[\sin y\right]_0^{\pi/2}$$
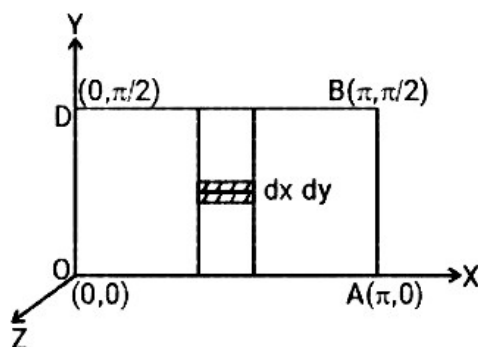
Fig.

Example : Use Green's theorem to evaluate :

$$\int_c \{(y - \sin x)dx + \cos x \, dy\}$$

where C is the triangle enclosed by the lines y = 0, x = π/2 and y = 2x/π.

Solution. Given P = y − sin x   and   Q = cos x

$$\therefore \quad \frac{\partial P}{\partial y} = 1 \quad \text{and} \quad \frac{\partial Q}{\partial x} = -\sin x$$

By Green's theorem,

$$\int_c (P \, dx + Q \, dy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy \qquad \dots(1)$$

Substituting the values of P and Q in (1) and simplifying

$$\int_c \left[(y - \sin x)dx + \cos x \, dy \right]$$

$$= \iint_S (-\sin x - 1) dx \, dy, \qquad \text{where S is the area of the triangle.}$$

$$= -\int_{x=0}^{\pi/2} \int_{y=0}^{2x/\pi} (1 + \sin x) dx \, dy$$

$$= -\int_0^{\pi/2} (1 + \sin x) \frac{2x}{\pi} dx$$

$$= -\frac{2}{\pi} \int_0^{\pi/2} (x + x \sin x) dx$$

$$= -\frac{2}{\pi} \left[\left(\frac{x^2}{2} \right)_0^{\pi/2} + \left\{ (-x \cos x)_0^{\pi/2} - \int_0^{\pi/2} 1(-\cos x) dx \right\} \right]$$

$$= -\frac{2}{\pi} \left[\frac{1}{2} \cdot \frac{\pi^2}{4} + (\sin x)_0^{\pi/2} \right]$$

$$= -\frac{2}{\pi} \left[\frac{\pi^2}{8} + 1 \right] = -\left[\frac{\pi}{4} + \frac{2}{\pi} \right]$$

Fig.
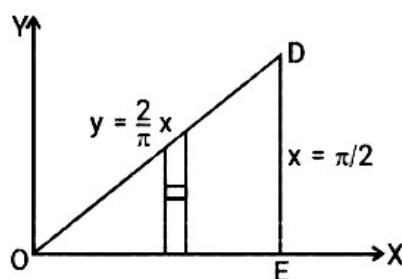
Example : Evaluate by Green's theorem :

$$\int_C \left(x^2 - \cosh y\right)dx + \left(y + \sin x\right)dy,$$

where C is the rectangle with vertices (0, 0), (π, 0), (π, 1) and (0, 1).

Solution. Taking the given points as A, B, C and D respectively, then a rectangle ABCD is obtained as given in the fig.

Here $P = x^2 - \cosh y \Rightarrow \dfrac{\partial P}{\partial y} = -\sinh y$

and $Q = y + \sin x \Rightarrow \dfrac{\partial Q}{\partial x} = \cos x$

Now by Green's theorem,

$$= \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\, dy$$

$$= \int_{x=0}^{\pi}\int_{y=0}^{1}\left(\cos x + \sinh y\right)dx\, dy$$

$$= \int_{x=0}^{\pi}\left[y\cos x + \cosh y\right]_0^1 dx$$

$$= \int_{x=0}^{\pi}\left(\cos x + \cosh 1 - 1\right)dx$$

$$= \left[\sin x + x\cosh 1 - x\right]_0^{\pi}$$

$$= \pi\cosh 1 - \pi = \pi[\cosh 1 - 1]$$

Fig.
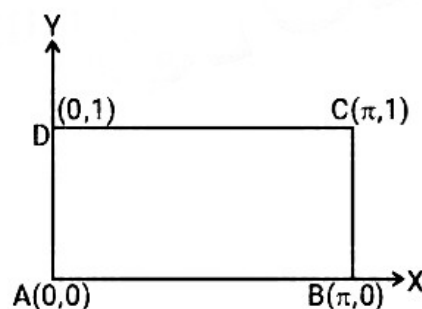
Example : Evaluate :

$$\int_C \left[\left(3x^2 - 8y^2\right)dx + \left(4y - 6xy\right)dy\right],$$

where C is the region bounded by the parabolas $y = \sqrt{x}$ and $y = x^2$. Also verify Green's theorem.

Solution. Solving the equations of the parabola $y = \sqrt{x}$ and $y = x^2$, the coordinates of the points of intersection are O(0, 0) and A(1, 1)

Here $P = 3x^2 - 8y^2$ and $Q = 4y - 6xy$

$\therefore$ $\dfrac{\partial P}{\partial y} = -16y$ and $\dfrac{\partial Q}{\partial x} = -6y$

By Green's theorem,

$$\iint_s\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\ dy$$

$$=\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}}\left\{-6y-(-16y)dx\ dy\right\}$$

$$=10\int_0^1\left[\frac{1}{2}y^2\right]_{x^2}^{\sqrt{x}}dx=5\int_0^1\left(x-x^4\right)dx$$

$$=\frac{3}{2}\qquad\qquad\qquad\qquad\qquad\qquad\text{...(1)}$$
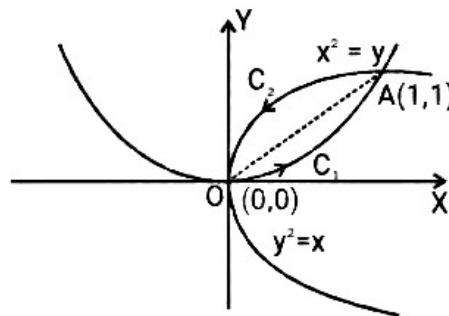
Fig.

Verification : The line integral towards C_1 (towards $y=x^2$)

$$\int_0^1\left\{\left[3x^2-8\left(x^2\right)^2\right]dx+\left(4x^2-6x\cdot x^2\right)2x\ dx\right\}=-1$$

and line integral towards C_2 (towards $y^2=x$)

$$\int_0^1\left(3x^2-8x\right)dx+\left(4\sqrt{x}-6x^{3/2}\right)\frac{1}{2}x^{-1/2}dx=\frac{5}{2}$$

Required integral = Line integral towards C_1 and C_2

$$=-1+\frac{5}{2}=\frac{3}{2}\qquad\qquad\qquad\qquad\qquad\text{...(2)}$$

Therefore from (1) and (2), the Green's theorem is verified.

Example : Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\int_c\left(x\,dy-y\,dx\right)$.

Hence find the whole area of the ellipse.

Solution. Here $P=-y$ and $Q=x$

Therefore $\frac{\partial P}{\partial y}=-1$ and $\frac{\partial Q}{\partial x}=1$

By Green's theorem, we know that

$$\int_c P\ dx+Q\ dy=\iint_s\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\ dy\qquad\qquad\text{...(1)}$$

Substituting the values of $\dfrac{\partial P}{\partial y}, \dfrac{\partial Q}{\partial x}$, P and Q in (1),

$$\int_c (-y\ dx + x\ dy) = \iint_s \left[1 - (-1)\right] dx\ dy$$

or $\quad \dfrac{1}{2}\int_c (x\ dy - y\ dx) = \iint_s dx\ dy$ = Required area $\qquad \qquad ...(2)$

Therefore the enclosed area by the simple closed curve C

$$= \dfrac{1}{2}\int_c (x\ dy - y\ dx)$$

The given curve is an ellipse whose parametric equations are

$\quad\quad x = a\ \cos\theta \quad\quad$ and $y = b\ \sin\theta.$

Therefore $x = a\ \cos\theta \quad$ and $y = b\ \sin\theta$

$\therefore\quad$ The required area of the ellipse

$$= \dfrac{1}{2}\int_0^{2\pi} \left(a\cos\theta\,\dfrac{dy}{d\theta} - b\sin\theta\,\dfrac{dx}{d\theta}\right) d\theta$$

$$= \dfrac{1}{2}\int_0^{2\pi} \left(ab\cos^2\theta - ab\sin^2\theta\right) d\theta$$

$$= \dfrac{1}{2}ab\int_0^{2\pi} d\theta = \pi\ ab$$

1. DIFFERENTIAL EQUATIONS

A relation connecting an independent variable x, a dependent variable y and one or more of their differential coefficients or differentials is called differential equation.

For example,

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 4)y = 0$$

A differential equation as given above which involves only one independent variable is called ordinary differential equations, while those involving more than one independent variables are called partial differential equations. Partial differential equations will involve partial derivatives.

For example,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \left(\frac{\partial z}{\partial x}\right)^3 + \frac{\partial z}{\partial y} = 0.$$

There are differential equations, which do not contain the variables explicitly.
For example,

$$\frac{dy}{dx} = 5, \qquad x\frac{d^2y}{dx^2} = 4.$$

A total differential equation contains two or more dependent variables together with their differentials or differential coefficients with respect to a single independent variable. This may or may not occur explicitly in the equation. In the total differential equation

u dx + v dy + w dz = 0,

where u, v, w are functions of x, y and z, any one of the variables may be regarded as independent and the others as dependent. Thus, taking x as independent variable, the above equation may be written as

$$u\frac{dx}{dx} + v\frac{dy}{dx} + w\frac{dz}{dx} = 0 \quad \text{or} \quad u + v\frac{dy}{dx} + w\frac{dz}{dx} = 0$$

The order of a differential equation is the order of the highest ordered differential coefficient involving in it while the degree of an equation is the greatest exponent of the highest ordered derivative when the equation has been made rational and integral as far as the derivatives are concerned.

Example

(i) $x^2 = y^2\left(\frac{dy}{dx}\right)^2$ is an equation of first order and second degree.

(ii) $\frac{dy}{dx} + xy = x^2$ is an equation of first order and first degree;

(iii) $\frac{d^2y}{dx^2} = \frac{x^2}{y(1 + \sqrt{x})}$ is an equation of second order and first degree;

(iv) $\left(\dfrac{d^2y}{dx^2}\right)^2$ = 4 is an equation of second order and second degree;

Let us consider the equation $\dfrac{d^2y}{dx^2} = 5\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$.

This equation is to be squared to rationalise it and then it can be easily seen that the greatest exponent of the highest ordered derivative $\dfrac{d^2y}{dx^2}$ is two. Hence the equation is of second order and second degree. It should be noted that the determination of the degree does not require the variables x and y to be made rational and integral.

When, in an ordinary or partial differential equation.

Note: In an ordinary or partial differential equations when the dependent variable and its derivatives occur to the first degree only, and not as higher powers or products, the equation is called linear; otherwise it is non-linear. The coefficients of a linear equation are therefore either constants of functions of the independent variable or variables. For examples,

$$\dfrac{d^2y}{dx^2} + y = x, \quad x^3\dfrac{d^2y}{dx^2} + (\cos x)\dfrac{dy}{dx} + (\sin x)\,y = 0$$

are ordinary linear differential equations of the second order while the equations

$$(x+y)^2\dfrac{dy}{dx} = a, \quad \dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + x(\sin y) = 0$$

are ordinary non-linear equations of the first and second order respectively.

Any relation connecting the dependent as well as the independent variables will be called the solution or primitive of the differential equation, if it reduces the differential to an identity when substituted in it. The solution of the differential equation does not contain any derivative. Thus

$$y = 2x \text{ is a solution of the differential equation } \dfrac{dy}{dx} = 2.$$

Formation of differential equations

Consider the relation y = cx, where c is an arbitrary constant.

Differentiating both sides of this, with respect to x, we get

$$\dfrac{dy}{dx} = c.$$

Eliminating the arbitrary constant from these two, we get the ordinary differential equation

$$\dfrac{dy}{dx} = \dfrac{y}{x}, \qquad\qquad\qquad ...(1)$$

which is of first order and first degree. Substituting y = cx in (1), we see that the equation is identically satisfied, showing that y = cx is a solution of the first equation (1). We notice further that the solution of a differential equation of first order and first degree will involve one arbitrary constant.

In a similar manner, differentiating both sides of the relation

$$y = A \cos (x + B), \qquad\qquad\qquad ...(2)$$

where A and B are arbitrary constants, twice with respect to x, we get

$$\frac{dy}{dx} = - A \sin (x + B) \qquad \qquad ...(3)$$

and $\qquad \dfrac{d^2y}{dx^2} = - A \cos (x + B) \qquad \qquad ...(4)$

Eliminating the arbitrary constants A and B from (2) and (4), we get the second ordered ordinary differential equation

$$\frac{d^2y}{dx^2} + y = 0 \qquad \qquad ...(5)$$

Thus (2) is a solution of the equation (5); for (2), when substituted in (5), reduces it to an identity. As before, we observe that the solution of a differential equation of second order involves two arbitrary constants.

The equations (1) and (5) may be written as

$$F\left(x,y,\frac{dy}{dx}\right) = 0 \text{ and } F\left(y,\frac{d^2y}{dx^2}\right) = 0 \text{ respectively,}$$

and their solutions are of the forms f(x, y, c) = 0 and
$\qquad$ f(x, y, A, B) = 0.
Consider now the general relation
$\qquad$ $f(x,\ y,\ c_1,\ c_2,....,c_n) = 0 \qquad \qquad ...(6)$
which contains, besides the variables x and y, n arbitrary constants
$\qquad$ $c_1,\ c_2,\ ....,\ c_n$
Differentiating (6) n times in succession, with respect to x, we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0,$$

$$\frac{\partial^2 f}{\partial x^2}+2\frac{\partial^2 f}{\partial x \delta y}\frac{dy}{dx}+\frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2+\frac{\partial f}{\partial y}\frac{d^2y}{dx^2}=0,$$

....
....

$$\frac{\partial^n f}{\partial x^n}+\text{............................}+\frac{\partial f}{\partial y}\frac{d^ny}{dx^n}=0.$$

Then, by eliminating the n arbitrary constants from these n equations and the original equation (6), we shall obtain the differential equation

$$F\left(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2},.....,\frac{d^ny}{dx^n}\right)=0,$$

of which (6) is the primitive. This is the typical linear differential equation of order n and degree one.

Thus, if we eliminate the n constants from a relation in x and y involving n arbitrary constants and the n equations arising from its n successive differentiation, then we get the corresponding differential equation, of which the assumed relation in x and y involving n arbitrary constants is the primitive. There being n differentiations, the resulting differential equation will be of n-th order and free from those arbitrary constants.

This shows that a solution of the n-th order ordinary differential equation should involve n arbitrary constants.

On the other hand, the n-th ordered differential equation cannot have more than n arbitrary constants in its primitive. For, if it had (n +1) arbitrary constants, on eliminating them, there would appear an equation of (n + 1)th order and not an equation of the n-th order.

General solution and arbitrary constants

Assuming the existence of the solution of a differential equation, we consider the differential equation $\frac{dy}{dx}$ = x. A solution of this equation is y = $\frac{1}{2}$x² and another solution is y = $\frac{1}{2}$x² + c, c being a constant, For different values given to c, we get different solutions and in particular, by giving c the value zero, the solution y = $\frac{1}{2}$x² is obtained.

The solutions of the equation $\frac{d^2y}{dx^2}$ + y = 0

are   y = cos x and y = sin x.

More general solutions are y = A cos x and y = B sin x, where A and B are constants. The former solutions are obtained from these general solutions by giving A and B the particular value unity. Yet more general solution of the equation is

y = A cos x + B sin x,

from which all the previous solutions are obtained by giving particular values to the arbitrary constants A and B.

The solution, which contains a number of arbitrary constants equal to the order of the differential equation, is called the general solution, the complete primitive or the complete integral. Solutions obtained from the general solution by giving particular values to the arbitrary constants are called particular solutions.

In some special cases, we get solutions of a differential equation which cannot be derived from its general solution by giving particular values to the arbitrary constants. These solutions are known as singular solutions.

In some linear differential equations, the general solution consists of terms involving the arbitrary constants and terms giving a definite function of the independent variable. The first part is called the complementary function and the remaining part, which can be obtained by giving the value zero to each of the arbitrary constants, is called the particular integral. Thus, if the complete primitive be given by

y = A cos x + B sin x + 3e^x,

then (A cos x + B sin x) is the complementary function and 3e^x is the particular integral.

Note 1. The solution of a differential equation involving arbitrary constants may be expressed in various forms. Thus

log cos x – log (2 – y) = c

can be written, on simplification, as y = 2 – k cos x.

Note 2. An arbitrary constant in the solution of a differential equation is said to be independent, if it be impossible to deduce from the solution an equivalent relation which will contain fewer arbitrary constants. Thus the two arbitrary constants a and c in the solution y = ae^{x + c} are not independent as they are really equivalent to only one; for,

y = ae^{x + c} = ae^x . e^c = Ae^x, where A = ae^c is another constant.

This A is independent.

Note 3. In our above discussion, we have assumed that the conditions, which ensure the existence of the solutions of a given differential equation, are satisfied.

Geometrical significance of differential equation.

Let us consider a differential equation

$$f\left(x, y, \frac{dy}{dx}\right) = 0 \qquad \qquad ...(1)$$

of first order and first degree. The primitive of this equation is a relation between two variables x and y and a parameter c, such as

$$\phi(x, y, c) = 0, \qquad \qquad ...(2)$$

which, for different numerical values attributed to c, represents different curves. Thus the differential equation represents a single parameter family of curves and each member of this family is called an integral curve.

Now, we know that in Cartesian co-ordinate system, $\frac{dy}{dx}$ of a curve gives the direction of the tangent to the curve at a given point. Let the value of $\frac{dy}{dx}$ at a point (x_1, y_1) as given by the equation (1) be m_1. Thus, as a point moves through (x_1, y_1) in the direction m_1 and reaches a point (x_2, y_2) at an infinitesimal distance, then at that point it will move in a direction m_2 as given by (x_2, y_2) and restricted by the equation (1). It will move through an infinitesimal distance to a point (x_3, y_3) and from there under the same condition to (x_4, y_4) and so on through successive points. Thus the point will describe a curve by its movements such that the co-ordinates of any point of the curve and also the value of $\frac{dy}{dx}$ there, will satisfy the differential equation (1). Similar consideration shows that the point will describe some other curve if it starts from some other point on the xy-plane and moves such that the co-ordinates of a point on that curve along with the direction in which it moves satisfy (1). Through every point on the xy-plane, there will pass a particular curve, for every point of which the co-ordinates x, y and the direction of the tangent $\frac{dy}{dx}$ will satisfy the differential equation (1). The equation of each curve is thus a particular solution of the differential equation, the whole system of curves being its general solution which in its totality makes the locus of the differential equation.

If the equation (1) be of second degree in $\frac{dy}{dx}$, then there will be two values of the direction for each particular point (x_1, y_1) and hence the point can move through this point in either of these directions and hence two curves of the system, which is the locus of the general solution, will pass through each point. The general solution (2) of the equation must be such that c in that solution must be of second degree.

Generalising this, we can say that a linear differential equation of order one and degree n must contain an arbitrary constant in its general solution whose degree must be n. n of these curves given by the general solution for a particular value of the parameter will pass through each point of the plane.

The general solution of a differential equation of second order

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \qquad \qquad ...(3)$$

contains two arbitrary constants and therefore the aggregate of integral curves will form a two-parameter family.

2. CAUCHY'S PROBLEM

OR

ODE OF THE FIRST ORDER OF THE FORM Y' = F(X, Y)

Consider the first ordered differential equation $\frac{dy}{dx}$ = f(x, y), whose general solution contains one arbitrary constant.

It is sufficient to specify the value y_0 of the particular solution for some value x_0 of the independent variable x, that is, to find a point (x_0, y_0) through which the integral curve of the given equation must pass.

But this is not sufficient for equations of higher order. For instance, the general solution of the equation $\frac{d^2y}{dx^2}$ = 0 is

$$y = Ax + B,$$

where A and B are arbitrary constants.

The equation y = Ax + B defines a two-parameter family of straight lines on the xy-plane. To specify a definite straight line, it is not sufficient to specify a point (x_0, y_0) through which the line must pass. It is also necessary to specify the slope of the line at the point (x_0, y_0) as given by

$\frac{dy}{dx}$ at x = x_0, that is , $\left(\frac{dy}{dx}\right)_0$.

In the general case of the n-th ordered differential equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots \frac{d^ny}{dx^n}\right) = 0, \qquad \dots(1)$$

in order to isolate a particular solution, we must have n conditions

$$y = y_0, \ \frac{dy}{dx} = \left(\frac{dy}{dx}\right)_0, \dots\dots\dots, \frac{d^{n-1}y}{dx^{n-1}} = \left(\frac{d^{n-1}y}{dx^{n-1}}\right)_0, \ \text{at } x = x_0 \qquad \dots(2)$$

in which y_0, $\left(\frac{dy}{dx}\right)_0$, $\dots\dots\dots, \left(\frac{d^{n-1}y}{dx^{n-1}}\right)_0$ are given numbers.

Cauchy's problem is to find a solution of the differential equation (1) which satisfies the specific conditions (2).

Example. Eliminate the arbitrary constants A and B from the relation.

$$y = Ae^x + Be^{-x} + x^2.$$

Sol. From the given relation, we have

$$\frac{dy}{dx} = Ae^x - Be^{-x} + 2x$$

and $\quad \frac{d^2y}{dx^2} = Ae^x + Be^{-x} + 2 = y - x^2 + 2.$

Therefore

$$\frac{d^2y}{dx^2} - y = 2 - x^2.$$

This is the required differential equation

Example. Show that the differential equation satisfied by the family of curves given by $c^2 + 2cy - x^2 + 1 = 0$, where c is the parameter of the family, is

$$(1 - x^2)\, p^2 + 2xyp + x^2 = 0, \text{ where } p = \frac{dy}{dx}.$$

Sol. From the given equation of the family of curves, we have, on differentiation with respect to x,

$$2cp - 2x = 0.$$

Therefore $\quad c = \dfrac{x}{p}.$

Eliminating c from the given equation, we get the corresponding differential equation as

$$\frac{x^2}{p^2} + \frac{2xy}{p} - x^2 + 1 = 0$$

or, $\quad (1 - x^2)\, p^2 + 2xyp + x^2 = 0.$

Example. Obtain the differential equation of the system of confocal conics

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1,$$

in which λ is the arbitrary parameter and a, b are given constants.

Sol. Let us eliminate λ from the given equation and its first derived equation

$$\frac{2x}{a^2 + \lambda} + \frac{2yy'}{b^2 + \lambda} = 0, \text{ where } y' = \frac{dy}{dx}.$$

From the last equation, we have

$$\frac{a^2 + \lambda}{x} = \frac{b^2 + \lambda}{-yy'} = k(\text{say}). \qquad \qquad \text{...(1)}$$

Putting this in the given equation, we have

$$\frac{x^2}{kx} + \frac{y^2}{-kyy'} = 1$$

or $\quad k = -\dfrac{1}{y}(y - xy').$

Substituting this value of k in (1), we get

$$a^2 + \lambda = kx = \frac{x^2y' - yx}{y'}$$

and $\quad b^2 + \lambda = -kyy' = y(y - xy').$

Subtracting, we have

$$a^2 - b^2 = \frac{x^2y' - yx}{y'} + (xy' - y)y$$

Therefore $\quad (a^2 - b^2)\, y' = x(xy' - y) + yy'\,(xy' - y)$

$$= (xy' - y)\,(x + yy').$$

This is the required differential equation, whose primitive is the given system of confocal conics.

Equations of First Order and First Degree

Given a differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \qquad ...(1)$$

where f(x, y) is subject to the following conditions :

(i)   f(x, y) is continuous in a given region G,

(ii) | f(x, y) | ≤ M, a fixed real number in G,

(iii) | f(x, y_1) – f(x, y_2) | ≤ k | y_1 – y_2 |, k being a fixed quantity for any two points (x, y_1) and (x, y_2) in the region G.

If now (x_0, y_0) be any point in G such that the rectangular region R as given by |x – x_0| ≤ a, |y – y_0| ≤ b, where b > aM such that R lies wholly within G, then there exists one and only one continuous function y = ϕ(x) having continuous derivatives in |x – x_0| ≤ a, which satisfies the differential equation (1) and takes up the value ϕ(x_0) = y_0 when

$$x = x_0.$$

The condition (iii), that is, |f(x, y_1) – f(x, y_2)| ≤ k |y_1 – y_2| provided (x, y_1) and (x, y_2) are any two points in G, is known as Cauchy-Lipschitz condition. If f(x, y) admits of continuous partial derivatives and

hence $\left| \dfrac{\partial}{\partial y} f(x, y) \right|$ < k, where k is fixed, then the Cauchy-Lipschitz condition is satisfied.

An ordinary differential equation of first order and first degree

$$\frac{dy}{dx} = f(x, y)$$

can always be written as M dx + N dy = 0,

Where M and N are functions of x and y. Assuming that the equation has a solution, we shall discuss methods by which the general solutions of these equations can be found in terms of known functions. We classify these equations according to the methods by which they are solved. These classifications are

(A)   Equations solvable by separation of variables,

(B)   Homogeneous equations,

(C)   Exact equations,

(D)   Linear equations.

Solution by separation of variables

If the equation M dx + N dy = 0 can be put in the form

f_1(x) dx + f_2(y) dy = 0,

that is, in the separated variables form, then the equation can be solved easily by integrating each term separately. The general solution of the above equation is

$$\int f_1(x)dx + \int f_2(x)dy = c,$$

where c is an arbitrary constant. By giving particular values to c, we shall get particular solutions. The equations of the form

$$\frac{dy}{dx} = f(x)\, g(y) \quad \text{or} \quad f_1(x)\, \phi_1(y)\, dx + f_2(y)\, \phi_2(x)\, dy = 0$$

can also be put in the above form and integrated. Sometimes a transformation of the dependent variable will be used to facilitate separation of variables.

Example. Solve : $x^2 \dfrac{dy}{dx} + y = 1$.

Sol. We have, from the given equation

$$x^2 \frac{dy}{dx} = 1 - y$$

or, $\qquad \dfrac{dx}{x^2} = \dfrac{dy}{1-y}$

The variables have been separated.

Now, integrating both sides, we get the general solution as

$$-\frac{1}{x} = -\log(1 - y) + c_1, \text{ where } c_1 \text{ is an arbitrary constant}$$

or, $\qquad \log \dfrac{1-y}{c} = \dfrac{1}{x}$, taking $c_1 = \log c$

or, $\qquad 1 - y = c\, e^{1/x}$

or, $\qquad y = 1 - c\, e^{1/x}$.

Example. Solve : $x\sqrt{y}\,dx + (1+y)\sqrt{1+x}\,dy = 0$.

Sol. Dividing throughout by $\sqrt{y(1+x)}$, we get

$$\frac{x\,dx}{\sqrt{1+x}} + \frac{1+y}{\sqrt{y}}\,dy = 0$$

or, $\qquad \left(\sqrt{1+x} - \dfrac{1}{\sqrt{1+x}}\right)dx + \left(\dfrac{1}{\sqrt{y}} + \sqrt{y}\right)dy = 0$

Integrating, we get the general solution as

$$\frac{2}{3}(1+x)^{\frac{3}{2}} - 2\sqrt{1+x} + 2\sqrt{y} + \frac{2}{3}y^{\frac{3}{2}} = c, \text{ where } c \text{ is an arbitrary constant}$$

or $\qquad (x - 2)\sqrt{1+x} + (y + 3)\sqrt{y} = k, \text{ where } k = \dfrac{3c}{2}$.

Example. (a) Solve the equation $(x + y)^2 \dfrac{dy}{dx} = a^2$.

(b) Show that the general solution of the equation $\dfrac{dy}{dx} + Py = Q$ can be written in the form

$y = k(u - v) + v$, where k is a constant and u and v are its two particular solutions.

    Sol. (a) In this case, we are to apply certain transformation of the dependent variable to have the variables separated.

We put $\qquad x + y = v$, so that $1 + \dfrac{dy}{dx} = \dfrac{dv}{dx}$.

Putting these in the equation, we get

$$v^2\left(\frac{dv}{dx}-1\right)=a^2$$

or, $\dfrac{dv}{dx}=\dfrac{a^2}{v^2}+1=\dfrac{a^2+v^2}{v^2}.$

Separating the variables, we get

$\dfrac{v^2}{a^2+v^2}$ dv = dx, that is, $\left(1-\dfrac{a^2}{a^2+v^2}\right)$ dv = dx.

Integrating both sides, we get $v-a\tan^{-1}\dfrac{v}{a}=x+c.$

Substituting (x + y) for v, we get

$$x+y-a\tan^{-1}\frac{x+y}{a}=x+c$$

or, $y=a\tan^{-1}\dfrac{x+y}{a}+c$, which is the general solution.

Note. To solve equations of the form $\dfrac{dy}{dx}=f(ax+by+c)$, the general substitution is **ax + by + c = v.**

(b) Since u and v are the solutions of $\dfrac{dy}{dx}+Py=Q,$ ...(1)

therefore $\dfrac{du}{dx}+Pu=Q$ ...(2)

and $\dfrac{dv}{dx}+Pv=Q$ ...(3)

Subtracting (3) from (2), we get $\dfrac{d}{dx}(u-v)+P(u-v)=0.$ ...(4)

Subtracting (3) from (1), we get $\dfrac{d}{dx}(y-v)+P(y-v)=0$ ...(5)

From (4) and (5), we have $\dfrac{d(u-v)}{u-v}=-P\ dx=\dfrac{d(y-v)}{y-v}.$

Integrating, we get log (y − v) = log(u − v) + log k, where k is a constant.

Thus   y − v = k(u − v), giving y = k (u − v) + v.

Example. The product of the slope and the ordinate at any point (x, y) of a curve passing through the point (5, 3) is equal to the abscissa at that point. Find the equation of the curve.

Find the equation of the curve.

Sol. The slope at any point (x, y) of a curve is $\dfrac{dy}{dx}$.

By the given condition, we have $y\dfrac{dy}{dx}=x$, that is, y dy = x dx.

Integrating, we get $x^2 - y^2 = c$, where c is an arbitrary constant.

Since the curve passes through the point (5, 3), we have

$c = 25 - 9 = 16$.

Therefore the required equation of the curve is $x^2 - y^2 = 16$.

Homogeneous equation.

If a function f(x, y) can be expressed in the form $x^n \phi\left(\dfrac{y}{x}\right)$ in the form $y^n \psi\left(\dfrac{x}{y}\right)$, then f(x, y) is said to be a homogeneous function degree n in x and y.

If the equation M dx + N dy = 0 can be put in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right),$$

If M and N be homogeneous functions of x and y of same degree, then the equation is called homogeneous. In this case, by y = vx, where v is a function of x, we can change the variable, such that

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Then, after integration, v is replaced by $\dfrac{y}{x}$.

Note. $\dfrac{dy}{dx} = f(x, y)$ is a homogeneous equation, if f(tx, ty) = f(x, y) for all t.

It can be easily verified that the equations

$$\frac{dy}{dx} = 3 \log (x + y) - \log (x^3 + y^3) \text{ and } \frac{dy}{dx} = \frac{4y\sqrt{x} - 5x\sqrt{y}}{\sqrt{x}(-x + 7y)} \text{ are homogeneous.}$$

Non-homogeneous equation reducible to homogeneous form

Consider a non-homogeneous equation of the form

$(a_1x + b_1y + c_1)dx = (a_2x + b_2y + c_2)dy,$

that is, $\dfrac{dy}{dx} = \dfrac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2},$ ...(1)

in which $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$.

This equation can be made homogeneous by the substitution

x = x' + h and y = y' + k,

where h and k are constants and are so chosen that

$a_1h + b_1k + c_1 = 0$ ...(2)

and $a_2h + b_2k + c_2 = 0$.

The relations (2) determine the constants h and k.

The equation (1) is then reduced to the homogeneous equation

$$\frac{dy'}{dx'} = \frac{a_1x' + b_1y'}{a_2x' + b_2y'}$$

which can be solved after the substitution y' = vx' as before.

If $\quad \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$, then the substitution

$$a_1 x + b_1 y = v,$$

that is, $a_1 + b_1 \dfrac{dy}{dx} = \dfrac{dv}{dx}$

will transform the equation to a form, which can be easily solved.

Example. Solve: $x \cos \dfrac{y}{x} (y\, dx + x\, dy) = y \sin \dfrac{y}{x} (x\, dy - y\, dx)$.

Sol. The given equation can be put as

$$\left(xy \cos \dfrac{y}{x} + y^2 \sin \dfrac{y}{x} \right) dx = \left(-x^2 \cos \dfrac{y}{x} + xy \sin \dfrac{y}{x} \right) dy$$

or $\qquad \dfrac{dy}{dx} = \dfrac{y \left(x \cos \dfrac{y}{x} + y \sin \dfrac{y}{x} \right)}{x \left(-x \cos \dfrac{y}{x} + y \sin \dfrac{y}{x} \right)}$

The equation being homogeneous, let us put y = vx, so that

$$\dfrac{dy}{dx} = v + x \dfrac{dv}{dx}.$$

Thus $\quad v + x \dfrac{dv}{dx} = \dfrac{v(\cos v + v \sin v)}{(-\cos v + v \sin v)}$

or, $\quad x \dfrac{dv}{dx} = v \left(\dfrac{\cos v + v \sin v}{v \sin v - \cos v} - 1 \right) = \dfrac{2v \cos v}{v \sin v - \cos v}$

or, $\quad \dfrac{v \sin v - \cos v}{v \cos v} dv = \dfrac{2\, dx}{x}.$

Integrating both sides, we get

$$\log \sec v - \log v = 2 \log x + \log c,$$

where c is a constant.

This gives $\dfrac{\sec v}{v} = cx^2$.

Putting $v = \dfrac{y}{x}$, we get the solution as $\sec \dfrac{y}{x} = cxy$.

Example. Solve : $x^2 y\, dx - (x^3 + y^3)dy = 0$.

Sol. Here M and N are homogeneous functions of x and y of degree 3.

The given equation can be written as

$$\dfrac{dy}{dx} = \dfrac{x^2 y}{x^3 + y^3}.$$

Let $y = vx$, so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

Then the equation becomes

$$v + x\frac{dv}{dx} = \frac{x^3 v}{x^3 + x^3 v^3} = \frac{v}{1 + v^3}$$

or, $\qquad x\dfrac{dv}{dx} = \dfrac{v}{1 + v^3} - v = -\dfrac{v^4}{1 + v^3}$

or, $\qquad \dfrac{1 + v^3}{v^4}dv = -\dfrac{dx}{x}$

in which the variables v and x have been separated.

This gives $\qquad \left(\dfrac{1}{v^4} + \dfrac{1}{v}\right)dv = -\dfrac{dx}{x}$.

Integrating both sides, we get

$$-\frac{1}{3v^3} + \log v = -\log x + \log c, \text{ c being a constant}$$

or, $\qquad \log\dfrac{vx}{c} = \dfrac{1}{3v^3}$.

Therefore $\qquad \dfrac{vx}{c} = e^{\frac{1}{3v^3}} = e^{\frac{x^3}{3y^3}}$.

Thus the solution is $\quad y = ce^{\frac{x^3}{3y^3}}$

Example. Solve : $(x + 2y - 3)\,dx = (2x + y - 3)\,dy$.

Sol. We have the equation $\dfrac{dy}{dx} = \dfrac{x + 2y - 3}{2x + y - 3}$

To reduce this to the homogeneous form, let us put $x = x' + h$ and $y = y' + k$,

so that $\dfrac{dy}{dx}$ becomes $\dfrac{dy'}{dx'}$.

Therefore the equation becomes $\dfrac{dy'}{dx'} = \dfrac{x' + 2y' + (h + 2k - 3)}{2x' + y' + (2h + k - 3)}$.

Let us choose h and k such that

$\qquad\qquad h + 2k - 3 = 0$

and $\qquad\qquad 2h + k - 3 = 0$.

These give $h = k = 1$ and the equation reduces to

$$\frac{dy'}{dx'} = \frac{x' + 2y'}{2x' + y'}$$

which is a homogeneous equation.

We now put y' = vx', so that $\dfrac{dy'}{dx'} = v + x' \dfrac{dv}{dx'}$

The equation now becomes

$$v + x'\frac{dv}{dx'} = \frac{1+2v}{2+v}$$

or, $$x'\frac{dv}{dx'} = \frac{1+2v}{2+v} - v = \frac{1-v^2}{2+v}$$

or, $$\frac{2+v}{1-v^2}dv = \frac{dx'}{x'}$$

or, $$\left\{\frac{2}{1-v^2} - \frac{1}{2}\frac{(-2v)}{1-v^2}\right\}dv = \frac{dx'}{x'}$$

Integrating both sides, we get

$$\log\frac{1+v}{1-v} - \frac{1}{2}\log(1-v^2) = \log(cx')$$

Putting $v = \dfrac{y'}{x'}$, we get

$$\log\frac{x'+y'}{x'-y'} - \frac{1}{2}\log\frac{x'^2 - y'^2}{x'^2} = \log(cx')$$

or, $$\frac{x'+y'}{x'-y'}\cdot\frac{x'}{\sqrt{x'^2 - y'^2}} = cx'$$

Putting x' = x − 1 and y' = y − 1 as h = k = 1, we get

$$\frac{x+y-2}{x-y}\cdot\frac{1}{\sqrt{(x-1)^2 - (y-1)^2}} = c$$

or, $(x + y - 2)^2 = c^2(x - y)^2\{(x - 1)^2 - (y - 1)^2\}$

or, $x + y - 2 = c^2(x - y)^3.$

3. EXACT EQUATION AND ITS SOLUTION BY INSPECTION

If the differential equation $M\,dx + N\,dy = 0$ can be expressed in the form $du = 0$, where u is a function of x and y, without multiplying by any factor, then the equation $M\,dx + N\,dy = 0$ is said to be an exact differential equation and its general solution is $u(x, y) = c$, where c is an arbitrary constant.

A few exact differentials are given below

$$x\,dy + y\,dx = d(xy)\ ;$$

$$\frac{x\,dy + y\,dx}{xy} = d[\log(xy)]$$

$$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)\ ;$$

$$\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)\ ;$$

$$\frac{x\,dy - y\,dx}{xy} = d\left(\log\frac{y}{x}\right)\ ;$$

$$\frac{y\,dx - x\,dy}{x^2 + y^2} = d\left(\tan^{-1}\frac{x}{y}\right)\ ;$$

$$\frac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)\ ;$$

$$\frac{2\,xy\,dy - y^2\,dy}{x^2} = d\left(\frac{y^2}{x}\right)\ ;$$

$$\frac{y^2\,dx + 2\,xy\,dy}{x^2\,y^4} = -d\left(\frac{1}{xy^2}\right)\ ;$$

$$\frac{x\,dy + y\,dx}{\sqrt{1 - x^2 y^2}} = d\,\{\sin^{-1}(xy)\}.$$

$$\frac{y\,dx - x\,dy}{xy} = d\left[\log\left(\frac{x}{y}\right)\right]$$

$$\frac{x\,dx + y\,dy}{x^2 + y^2} = d\left[\log\sqrt{x^2 + y^2}\right]$$

$$\frac{x\,dx + y\,dy}{x^2 y^2} = d\left(-\frac{1}{xy}\right);$$

$$\frac{2\,yx\,dx - x^2\,dy}{y^2} = d\left(\frac{x^2}{y}\right)$$

$$\frac{2x^2 y\,dy - 2xy^2 dx}{x^4} = d\left(\frac{y^2}{x^2}\right);$$

$$\frac{2y^2 x\,dx - 2yx^2 dy}{y^4} = d\left(\frac{x^2}{y^2}\right)$$

$$\frac{ye^x dx - e^x dy}{y^2} = d\left(\frac{e^x}{y}\right);$$

$$\frac{xe^y dy - e^y dx}{x^2} = d\left(\frac{e^y}{x}\right)$$

$$2\left(\frac{x\,dy - y\,dx}{x^2 - y^2}\right) = d\left[\log\frac{x + y}{x - y}\right];$$

$$\frac{x\,dy - y\,dx}{x\sqrt{(x^2 - y^2)}} = d\left[\sin^{-1}\left(\frac{y}{x}\right)\right]$$

$$-2(n - 1)\left[\frac{x\,dy + y\,dx}{(x^2 + y^2)^n}\right] = d(x^2 + y^2)^{-n}, n \neq 1$$

$$-(n - 1)\left[\frac{x\,dy + y\,dx}{(xy)^n}\right] = d\left[\frac{1}{(xy)^{n-1}}\right]^{n+1} n \neq 1$$

General method of solution of exact equations

We have stated earlier that the ordinary differential equation $M\,dx + N\,dy = 0$ will be exact, if there exists a function $u(x, y)$, such that $M\,dx + N\,dy = du$. We now establish the condition for that.

Theorem. The necessary and sufficient condition for the ordinary differential equation $M\,dx + N\,dy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

[We assume here that the functions M and N have continuous partial derivatives.]

Proof: If the equation M dx + N dy = 0 ...(1)

be exact, then there must be a function u of x and y, such that

M dx + N dy = du, ...(2)

a total differential of u = du.

Also, we have $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$, ...(3)

x and y being independent variables.

Now the two expressions (2) and (3) for du are identical and hence, from (2) and (3), we shall have

$$\frac{\partial u}{\partial x} = M \text{ and } \frac{\partial u}{\partial y} = N.$$

Therefore $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$ and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}$

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,

if the partial derivatives of M and N be continuous.

Thus the condition is necessary.

To prove that this condition is also sufficient, we have to show that

if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then M dx + N dy = du.

Let us put P = $\int$ M dx, where, in the integrand, y is supposed to be a constant.

Then $\frac{\partial P}{\partial x} = M$ and we have

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial^2 P}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right).$$

Therefore N = $\frac{\partial P}{\partial y}$ + f(y), where f(y) is a function of y.

Using these values of M and N, we can write

$$M dx + N dy = \frac{\partial P}{\partial x}dx + \left\{\frac{\partial P}{\partial y} + f(y)\right\}dy$$

$$= d\{P + F(y)\}, \text{ where } dF(y) = f(y) \, dy.$$

Now, writing P + F(y) = u(x, y), we have

M dx + N dy = du.

Thus, to solve an equation of the from M dx + N dy = 0, we have to arrange the terms in groups each of the which is an exact differential, so that u(x, y) may be obtained by inspection only. This method has been discussed earlier.

If this cannot be done, then we have to test the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for the exactness of the equation first.

If it is found to be exact then, to determine the function u(x, y), we use the relation

$$\frac{\partial u}{\partial x} = M,$$

which on integration gives u = $\int$ M dx + f(y), where f(y) is a function of y.

Now, to determine the function f(y), we equate the total differential of u(x, y) to (M dx + N dy).

We see that all the terms of u(x, y) containing x must appear in $\int$ M dx. Hence the differential of this integral with respect to y must have all terms of N dy which contain x. Hence the rule for solving an exact equation of the form M dx + N dy = 0 is

Integrate the terms of M dx considering y as constant ; then integrate those terms of N dy which do not contain x and then equate the sum of these integrals to a constant.

Cor. In the exact equation M dx + N dy = 0, if M and N be homogeneous functions of x and y of degree n (≠ −1), then the primitive can be obtained without any integration and the primitive is

Mx + Ny = constant.

Proof: Since the equation is exact, we have $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. ...(1)

Again, since M and N are homogeneous functions of degree n, we have by Euler's theorem,

$$x\frac{\partial M}{\partial y} + y\frac{\partial M}{\partial y} = nM \qquad \qquad ...(2)$$

and

$$x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y} = nN \qquad \qquad ...(3)$$

Let u = Mx + Ny, so that we have

$$\frac{\partial u}{\partial x} = M + x\frac{\partial M}{\partial x} + y\frac{\partial N}{\partial x} = M + x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y} \text{, by (1)}$$

= M + nM, by (2)

= (n + 1) M.

Similarly, by (1) and (3), we get

$$\frac{\partial u}{\partial y} = (n + 1) \text{ N.}$$

Therefore $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$ = (n + 1) (M dx + N dy).

Hence M dx + N dy = $\frac{1}{n+1}$ du = $\frac{1}{n+1}$ d(Mx + Ny), n ≠ −1.

Thus the primitive is Mx + Ny = constant.

4. INTEGRATING FACTORS

Sometimes it is seen that an equation, as it stands is not exact but it can be made exact by multiplying it by some function of x and y. The function, which is multiplied to the equation to make it exact, is called integrating factor.

Let M dx + N dy = 0 be an ordinary differential equation and

$$Mx \pm Ny = 0.$$

We have $\dfrac{dy}{dx} = -\dfrac{M}{N} = \pm \dfrac{y}{x}$,

which can be integrated easily and in this situation no integrating factor is necessary.

Theorem. The number of integrating factors of an equation M dx + N dy = 0, which has a solution, is infinite.

Proof: Let μ(x, y) be an integrating factor of the equation M dx + N dy = 0,

so that   μ(M dx + N dy) = du.

Hence   u(x, y) = c is a solution of the equation.

If f(u) be any function of u, then

μf(u) (M dx + N dy) = f(u) du.

Now, the right-hand expression is an exact differential, since f(u) du can easily be integrated to give ϕ(u). Thus the solution of the equation is

ϕ(u) = c,

showing that μf(u) is also an integrating factor of the equation

M dx + N dy = 0.

Since f(u) is an arbitrary function of u, the number of integrating factors are infinite.

Rules for finding integrating factors

It is seen that an integrating factor can be found by inspection in simple cases. But in most cases when integrating factor cannot be found by inspection, the following rules are used to find it. For that, we consider the differential equation M dx + N dy = 0, in which

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Rule I. If Mx + Ny ≠ 0 and the equation be homogeneous, then $\dfrac{1}{Mx + Ny}$ is an integrating factor of the equation M dx + N dy = 0.

Proof: We have M dx + N dy

$$= \frac{1}{2}\left\{ (Mx + Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right) \right\}$$

$$= \frac{1}{2}\left[(Mx + Ny)\,d\{\log(xy)\} + (Mx - Ny)\,d\left(\log\frac{x}{y}\right) \right] \qquad ...(1)$$

Since Mx + Ny ≠ 0, we have, dividing both sides by (Mr + Ny),

$$\frac{M\,dx + N\,dy}{Mx + Ny} = \frac{1}{2}d\{\log(xy)\} + \frac{1}{2}\frac{Mx - Ny}{Mx + Ny}d\left(\log\frac{x}{y}\right) \qquad ...(2)$$

Now (Mx + Ny) is homogeneous; hence $\dfrac{Mx - Ny}{Mx + Ny}$ is also homogeneous, and is equal to a function

of $\dfrac{x}{y}$, say $f\left(\dfrac{x}{y}\right)$.

Therefore (2) becomes

$$\frac{M\,dx + N\,dy}{Mx + Ny} = \frac{1}{2} d\{\log(xy)\} + \frac{1}{2} f\left(\frac{x}{y}\right) d\left(\log \frac{x}{y}\right)$$

$$= \frac{1}{2} d\{\log(xy)\} + \frac{1}{2} F\left(\log \frac{x}{y}\right) d\left(\log \frac{x}{y}\right) \qquad \ldots(3)$$

since $\qquad f\left(\dfrac{x}{y}\right) = f\left\{e^{\log \frac{x}{y}}\right\} = F\left(\log \dfrac{x}{y}\right).$

The right-hand side of (3) is an exact differential.

Hence we see that $\dfrac{1}{Mx + Ny}$ is an integrating factor of the equation.

Rule II. If Mx − Ny ≠ 0 and the equation can be written as

$$\{f(xy)\} y\,dx + \{F(xy)\} x\,dy = 0,$$

then $\dfrac{1}{Mx - Ny}$ is an integrating factor of the equation.

Proof: Since Mx − Ny ≠ 0, dividing both sides of (1) by (Mx − Ny), we get

$$\frac{M\,dx + N\,dy}{Mx - Ny} = \frac{1}{2} \frac{Mx + Ny}{Mx - Ny} d\{\log(xy)\} + \frac{1}{2} d\left(\log \frac{x}{y}\right).$$

Now, we have M = {f(xy)} y and N = {F(xy)}x.
Therefore

$$\frac{M\,dx + N\,dy}{Mx - Ny} = \frac{1}{2} \frac{f(xy) + F(xy)}{f(xy) - F(xy)} d\{\log(xy)\} + \frac{1}{2} d\left(\log \frac{x}{y}\right)$$

$$= \frac{1}{2} \phi(xy) d\{\log(xy)\} + \frac{1}{2} d\left(\log \frac{x}{y}\right)$$

$$= \frac{1}{2} \psi\{\log(xy)\}\, d\{\log(xy)\} + \frac{1}{2} d\left(\log \frac{x}{y}\right) \qquad \ldots(4)$$

since $\phi(xy) = \phi\{e^{\log(xy)}\} = \psi\{\log(xy)\}$.

The right-hand expression of (4) being an exact differential, $\dfrac{1}{Mx - Ny}$ is an integrating factor of the equation.

Note. If Mx − Ny = 0 identically, then $\dfrac{M}{N} = \dfrac{y}{x}$ and the equation M dx + N dy = 0 reduces to x dy + y dx = 0, whose solution is xy = c.

Rule III. If $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ be a function of x along, say f(x), then $e^{\int f(x)dx}$ is an integrating factor of the equation.

Proof: Let μ be an integrating factor of the equation M dx + N dy = 0,

so that　　　(μM) dx + (μN) dy = 0

is an exact differential equation.

Hence the condition $\dfrac{\partial}{\partial y}$ (μM) = $\dfrac{\partial}{\partial x}$ (μN) must be satisfied.

This gives　　$M\dfrac{\partial \mu}{\partial y} - N\dfrac{\partial \mu}{\partial x} + \mu\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = 0$ 　　　...(5)

Now suppose μ is a function of x only, so that $\dfrac{\partial \mu}{\partial y}$ = 0.

Then (5) gives

$$\dfrac{d\mu}{\mu} = \dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)dx.$$ 　　　...(6)

Now, since μ is a function of x alone, the right hand side of (6) is a function of x only.

Let us put $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ = f(x),

so that (6) becomes $\dfrac{d\mu}{\mu}$ = f(x) dx,

which gives　　　μ = $e^{\int f(x)dx}$

Thus $e^{\int f(x)dx}$ is an integrating factor of the equation.

Note. $e^{\int P\,dx}$ is an integrating factor of the equation $\dfrac{dy}{dx}$ + Py = Q, where P and Q are functions of x only, since the equation can be written as (Py − Q) dx + dy = 0, so that M = Py − Q, N = 1 and $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ = P, which is a function of x alone.

Rule IV. If $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$ be a function of y alone, say ϕ(y), then $e^{\int \phi(y)dy}$ is an integrating factor of the equation.

Proof: The proof is similar to that in Rule III.

Rule V. If the equation be of the form

$x^a y^b$ (my dx + nx dy) = 0 ; a, b, m, n being constants, then $x^{km-a-1} y^{kn-b-1}$, where k has any value, is an integrating factor of the equation.

Proof: Let us assume that $x^p y^q$ is an integrating factor of the equation

$x^a y^b$ (my dx + nx dy) = 0.

Now $x^{p+a} y^{q+b}$ (my dx + nx dy) is an exact differential,

if $(m x^{p+a} y^{q+b+1} dx + n x^{p+a+1} y^{a+b} dy)$ be an exact differential.

This gives $m(q + b + 1) = n(p + a + 1)$

or, $\dfrac{q+b+1}{n} = \dfrac{p+a+1}{m} = k$, say where k is any number.

Therefore p = km − a − 1 and q = kn − b − 1.

Thus we see that $x^{km-a-1} y^{kn-b-1}$, where k is any number, is an integrating factor of the equation

$x^a y^b = $ (my dx + nx dy) - 0.

In this connection, it should be observed that $\dfrac{1}{k} x^{km} y^{kn}$, k ≠ 0 is the integral of the exact differential

$x^{km-1} y^{kn-1}$ (my dx + nx dy).

If the equation can be put in the form

$x^{a_1} y^{b_1} (m_1 y\, dx + n_1 x\, dy) + x^{a_2} y^{b_2} (m_2 y\, dx + n_2 x\, dy) = 0,$

then a factor, that will make the first term an exact differential is

$x^{k_1 m_2 - a_1 - 1} y^{k_1 n_1 - b_1 - 1}$

and a factor, that will make the second term an exact differential is

$x^{k_2 m_2 - a_2 - 1} y^{k_2 n_2 - b_2 - 1}$, where k_1 and k_2 have any value.

These two factors are identical, if

$k_1 m_1 - a_1 = k_2 m_2 - a_2$ and $k_1 n_1 - b_1 = k_2 n_2 - b_2.$

These easily determine k_1 and k_2, provided $m_1 n_2 - m_2 n_1 \neq 0$.

Example. Solve : $\dfrac{dy}{dx} + \dfrac{ax+hy+g}{hx+by+f} = 0$.

Sol. The given equation is

(hx + by + f)dy + (ax + hy + g)dx = 0

or,   ax dx + by dy + h(x dy + y dx) + g dx + f dy = 0

or, $a\, d\left(\dfrac{x^2}{2}\right) + b\, d\left(\dfrac{y^2}{2}\right) + h\, d\,(xy) + g\, dx + f\, dy = 0$

or, $d\left\{\dfrac{1}{2}(ax^2 + by^2) + h\,(xy) + gx + fy\right\} = 0.$

Integrating, we get

$\dfrac{1}{2}(ax^2 + by^2) + hxy + gx + fy + c = 0$, where c is an arbitrary constant.

Example. Solve : $(1 - x^2) \dfrac{dy}{dx} - 2xy = x - x^3$.

Sol. The equation can be written as

$(1 - x^2)\, dy - 2xy\, dx = x\, dx - x^3\, dx$

or, $d\{(1 - x^2)y\} = d\left(\dfrac{x^2}{2} - \dfrac{x^4}{4}\right)$

Integrating both sides, we get

$$y(1 - x^2) = \dfrac{1}{2}x^2 - \dfrac{1}{4}x^4 + c.$$

OR

$$(1 - x^2)\dfrac{dy}{dx} - 2xy = x - x^3$$

Dividing above equation by $(1 - x^2)$.

$$\dfrac{dy}{dx} - \dfrac{2xy}{1 - x^2} = \dfrac{x(1 - x^2)}{(1 - x^2)}$$

$$\dfrac{dy}{dx} - \dfrac{2x}{1 - x^2}.y = x \qquad \text{..........(1)}$$

Here $P = \dfrac{-2x}{1 - x^2}, \qquad Q = x$

$\text{I.F.} = e^{\int P dx} = e^{\int \frac{-2x}{1-x^2} dx} = e^{\log_e (1-x^2)} = (1 - x^2)$

Now using this formula, we get

$$y(\text{I.F.}) = \int Q \times \text{I.F.}\,dx + C$$

$$y(1 - x^2) = \int x.(1 - x^2)dx + C$$

$$y(1 - x^2) = \int (x - x^3)\,dx + C$$

$$y(1 - x^2) = \dfrac{x^2}{2} - \dfrac{x^4}{4} + C$$

Example. Solve $x\, dx + y\, dy + \dfrac{x\, dy - y\, dx}{x^2 + y^2} = 0$, given that $y = 1$ when $x = 1$.

Sol. The given equation can be put as

$$d\left(\dfrac{x^2}{2}\right) + d\left(\dfrac{y^2}{2}\right) + d\left(\tan^{-1}\dfrac{y}{x}\right) = 0$$

Integrating, we get

$$\dfrac{1}{2}(x^2 + y^2) + \tan^{-1}\dfrac{y}{x} + c = 0 \qquad \text{...(1)}$$

where c is an arbitrary constant.

Now, it is given that y = 1 when x = 1.

Putting these in (1), we get $1 + \tan^{-1} 1 + c = 0$, giving $c = -\dfrac{\pi}{4}$

Therefore the required particular solution is

$$\frac{1}{2}(x^2 + y^2) + \tan^{-1}\frac{y}{x} = 1 + \frac{\pi}{4}.$$

Example. Examine whether the equation
$$(a^2 - 2xy - y^2)dx - (x + y)^2\, dy = 0$$
is exact. If it be exact, then find the primitive.

Sol. Here $M = a^2 - 2xy - y^2$ and $N = -(x + y)^2$.

We have $\dfrac{\partial M}{\partial y} = -2x - 2y$ and $\dfrac{\partial N}{\partial x} = -2(x + y)$.

Hence the equation is exact.

The primitive of the equation is

$$\int (a^2 - 2xy - y^2)dx + \int (-y^2)\, dy = 0,$$

    y is considered as constant in the first integral

or, $a^2x - x^2y - xy^2 - \dfrac{1}{3}y^3 = c.$

Note. The first integral is $\displaystyle\int M\, dx$ and the second integral is

$$\int (\text{terms not containing x in N})\, dy.$$

Example. Solve the equation $4x^3y\, dx + (x^4 + y^4)\, dy = 0.$

Sol. Here $\dfrac{\partial M}{\partial y} = 4x^3 = \dfrac{\partial N}{\partial x}$. Hence the equation is exact.

Furthermore the equation is homogeneous.

Therefore the primitive is Mx + Ny = constant,

that is, $5x^4y + y^5 = c$, where c is a constant,

Example. Solve : $(x^2y - 2xy^2)\, dx + (3x^2y - x^3)\, dy = 0.$

Sol. We see that $\dfrac{\partial}{\partial y}(x^2y - 2xy^2) = x^2 - 4xy$

and $\dfrac{\partial}{\partial x}(3x^2y - x^3) = 6xy - 3x^2$

Therefore the equation is not exact but the coefficients of dx and dy are homogeneous. Hence

$$\frac{1}{(x^2y - 2xy^2)x + (3x^2y - x^3)y} = \frac{1}{x^2y^2}$$ will be an integrating factor.

Multiplying both sides of the equation by $\dfrac{1}{x^2y^2}$, we get

$$\frac{1}{y}dx - \frac{2}{x}dx + \frac{3}{y}dy - \frac{x}{y^2}dy = 0$$

or, $$\frac{y\,dx - x\,dy}{y^2} - 2\,d(\log x) + 3\,d(\log y) = 0$$

or, $$d\left(\frac{x}{y}\right) + d(3\log y - 2\log x) = 0$$

Therefore integrating, we get the primitive as

$$\frac{x}{y} + 3\log y - 2\log x = c,$$

where c is a constant.

Example. Solve : (xy sin xy + cos xy) y dx + (xy sin xy – cos xy) x dy = 0.

Sol. It is seen that the equation is not exact as $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ but M and N are of the forms

{f(xy)}y and {F(xy)}x.

Moreover Mx – Ny = 2xy cos xy.

Hence an integrating factor is $\frac{1}{2xy\cos xy}$.

Multiplying both sides of the given equation with this integrating factor, we get

$$\tan xy\,(y\,dx + x\,dy) + \frac{dx}{x} - \frac{dy}{y} = 0$$

or, tan xy d (xy) + d(log x) – d(log y) = 0.

Integrating, we get log sec xy + log x – log y = log c.

Hence $\frac{x}{y}$ sec xy = c is the primitive, where c is a constant.

Example. Solve : $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$.

Here $\frac{\partial M}{\partial y}$ = $12x^2y^3 + 2x$ and $\frac{\partial N}{\partial x}$ = $6x^2y^3 - 2x$. So the equation is not exact;

but $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{xy\,(3xy^3 + 2)}(6x^2y^3 - 2x - 12x^2y^3 - 2x) = -\frac{2}{y}$,

which is a function of y only.

Hence $e^{-\int \frac{2}{y}dy} = e^{-2\log y} = \frac{1}{y^2}$ is an integrating factor of the equation.

Multiplying both sides of the equation by $\frac{1}{y^2}$, we get

$$3x^2y^2dx + 2\frac{x}{y}dx + 2x^3y\,dy - \frac{x^2}{y^2}dy = 0$$

or, $\quad d(x^3y^2) + \dfrac{2xy\,dx - x^2dy}{y^2} = 0$

or, $\quad d(x^3y^2) + d\left(\dfrac{x^2}{y}\right) = 0$

Integrating, we get

$x^3y^2 + \dfrac{x^2}{y} = c$, c being a constant.

Example. Solve the equation

$(2x^2y - 3y^4)\,dx + (3x^3 + 2xy^3)\,dy = 0$.

Sol. This equation can be put as

$x^2(2y\,dx + 3x\,dy) + y^3(-3y\,dx + 2x\,dy) = 0$.

Here we have

$a_1 = 2$, $b_1 = 0$, $m_1 = 2$, $n_1 = 3$, $a_2 = 0$, $b_2 = 3$, $m_2 = -3$, $n_2 = 2$.

A factor, that will make the first term an exact differential is $x^{2k_1-2-1}y^{3k_1-1}$ and a factor, that will make the second term an exact differential, is $x^{-3k_2-1}y^{2k_2-3-1}$, where k_1 and k_2 have values such that these two factors are same. This gives $2k_1 - 3 = -3k_2 - 1$ and $3k_1 - 1 = 2k_2 - 4$, that is, $2k_1 + 3k_2 - 2 = 0$ and $3k_1 - 2k_2 + 3 = 0$.

These give $k_1 = -\dfrac{5}{13}$, $k_2 = \dfrac{12}{13}$. Thus the common factor is $x^{\frac{49}{13}}y^{\frac{28}{13}}$.

Multiplying by this integrating factor, the equation becomes

$$\left(2x^{\frac{23}{13}}y^{\frac{15}{13}} - 3x^{\frac{49}{13}}y^{\frac{24}{13}}\right)dx + \left(3x^{\frac{10}{13}}y^{\frac{28}{13}} + 2x^{\frac{36}{13}}y^{\frac{11}{13}}\right)dy = 0.$$

Here we see that $\quad \dfrac{\partial M}{\partial y} = -\dfrac{30}{13}x^{\frac{23}{13}}y^{-\frac{28}{23}} - \dfrac{72}{13}x^{\frac{49}{13}}y^{\frac{11}{13}}$

and $\quad \dfrac{\partial N}{\partial x} = -\dfrac{30}{13}x^{\frac{23}{13}}y^{-\frac{28}{13}} - \dfrac{72}{13}x^{\frac{49}{13}}y^{\frac{11}{13}}$

Therefore the equation is now exact.

Its primitive is

$$2\left(-\dfrac{13}{10}\right)x^{\frac{10}{13}}y^{\frac{15}{13}} - 3\left(-\dfrac{13}{36}\right)x^{\frac{36}{13}}y^{\frac{24}{13}} = c_1, \qquad c_1 \text{ being a constant}$$

or, $\quad -\dfrac{13}{5}x^{\frac{10}{13}}y^{\frac{15}{13}} + \dfrac{13}{12}x^{\frac{36}{13}}y^{\frac{24}{13}} = c_1$

or, $\quad 5x^{\frac{36}{13}}y^{\frac{24}{13}} - 12x^{\frac{10}{13}}y^{\frac{15}{13}} = c$, where $c = 60c_1$, c is another constant.

5. LINEAR EQUATION

An equation of the form

$$\frac{dy}{dx} + Py = Q, \qquad\qquad ...(1)$$

where P and Q are functions of x only (or constants) is called a linear equation of first order in y. The dependent variable and also its derivative in such equations occur in the first degree only and not as higher powers or products.

If both P and Q be constants, then the variables can be easily separated. This will also happen if either P or Q be zero.

Let R be an integrating factor of the above equation. Then the left hand side of the equation

$$R\frac{dy}{dx} + RPy = RQ$$

is the differential coefficient of some product. Now the first term $R\dfrac{dy}{dx}$ can only be derived by differentiating Ry.

We put

$$R\frac{dy}{dx} + RPy = \frac{d}{dx}(Ry) = R\frac{dy}{dx} + y\frac{dR}{dx}.$$

Therefore $RP = \dfrac{dR}{dx}$.

Integrating, we get $\log R = \int P\,dx$, so that $R = e^{\int P\,dx}$ is an integrating factor.

Now, multiplying both sides of the equation (1) by this integrating factor, we get

$$\frac{dy}{dx}e^{\int P\,dx} + Pye^{\int P\,dx} = Qe^{\int P\,dx}$$

or, $\qquad d\left(ye^{\int P\,dx}\right) = Qe^{\int P\,dx}dx.$

On integrating both sides, we get the primitive as

$$ye^{\int p\,dx} = \int Qe^{\int P\,dx}dx + c, \text{ where c is a constant.}$$

An alternative form of the solution of the equation (1).

Let $\quad z = \dfrac{Q}{P} - y$, that is, $Pz = Q - Py$.

Then $\quad \dfrac{dz}{dx} = \dfrac{d}{dx}\left(\dfrac{Q}{P}\right) - \dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{Q}{P}\right) - (Q - Py)$, from (1)

$$= \frac{d}{dx}\left(\frac{Q}{P}\right) - Pz, \text{ from (2).}$$

Therefore $\dfrac{dz}{dx} + Pz = \dfrac{d}{dx}\left(\dfrac{Q}{P}\right)$, which is linear in z.

The integrating factor of this equation is $e^{\int P dx}$. Multiplying the equation by the integrating factor $e^{\int P dx}$, we get

$$\frac{d}{dx}\left(ze^{\int P\,dx}\right) = e^{\int P\,dx}\frac{d}{dx}\left(\frac{Q}{P}\right).$$

On integration, we get $ze^{\int P\,dx} = \int e^{\int P\,dx}\frac{d}{dx}\left(\frac{Q}{P}\right)dx + c.$

Putting for z, we get an alternative form for the solution of the equation (1) as

$$\frac{Q}{P} - y = e^{-\int P dx}\left[\int e^{\int P\,dx}d\left(\frac{Q}{P}\right) + c\right]$$

or, $\quad y = \frac{Q}{P} - e^{-\int P dx}\left[\int e^{\int P\,dx}d\left(\frac{Q}{P}\right) + c\right]$

Note. Sometimes an equation may be linear in x, where y is the independent variable. The form of such an equation is

$$\frac{dx}{dy} + P_1 x = Q_1.$$

Here P_1 and Q_1 are functions of y only (or constants).
The general solution, in this case, will be

$$xe^{\int P_1\,dy} = \int Q_1 e^{\int P_1\,dy}dy + c$$

6. EQUATION REDUCIBLE TO LINEAR FORM OR BERNOULLI'S EQUATION

Let us consider the equation $\dfrac{dy}{dx} + Py = Qy^n$, which is known as Bernoulli's equation, in which P and Q are functions of x alone or constants.

It can be put ax $y^{-n} \dfrac{dy}{dx} + Py^{1-n} = Q$.

This equation may be brought to the linear form by the substitution

$y^{1-n} = v$, so that $\dfrac{dv}{dx} = (1-n)\, y^{-n} \dfrac{dy}{dx}$.

Thus the equation transforms to

$\dfrac{dv}{dx} + (1-n)P\, v = (1-n)\, Q,$

which is linear in v, its integrating factor being $e^{(1-n)\int P\, dx}$

The solution is given by

$$ v\, e^{(1-n)\int P\, dx} = (1-n)\int Q\, e^{(1-n)\int P\, dx}\, dx + c. $$

Then we put y^{1-n} for v.

Example. Solve : $1 + y^2 + \left(x - e^{-\tan^{-1} y}\right)\dfrac{dy}{dx} = 0$.

Sol. The equation can be written as

$$ \dfrac{dx}{dy} + \dfrac{x}{1+y^2} = \dfrac{e^{-\tan^{-1} y}}{1+y^2}, $$

which is linear in x.

Here $P = \dfrac{1}{1+y^2}$ and $Q = \dfrac{e^{-\tan^{-1} y}}{1+y^2}$.

Integrating factor is

$$ e^{\int P\, dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y} $$

Multiplying both sides of the equation by this integrating factor, we get

$$ \dfrac{dx}{dy} e^{\tan^{-1} y} + \dfrac{e^{\tan^{-1} y}}{1+y^2} x = \dfrac{1}{1+y^2} $$

or, $\quad d\left(x e^{\tan^{-1} y}\right) = \dfrac{1}{1+y^2}\, dy.$

Integrating both sides, we get the general solution as

$x e^{\tan^{-1} y} = \tan^{-1} y + c.$

Example. Solve : $\dfrac{dy}{dx} + y \cos x = y^n \sin 2x$.

Sol. We divide both sides of the equation by y^n and we get,

$$y^{-n} \dfrac{dy}{dx} + y^{1-n} \cos x = \sin 2x$$

Put $y^{1-n} = v$, so that $(1-n)\, y^{-n} \dfrac{dy}{dx} = \dfrac{dv}{dx}$ and the equation becomes

$$\dfrac{1}{1-n} \dfrac{dv}{dx} + v \cos x = \sin 2x$$

or, $\qquad \dfrac{dv}{dx} + (1-n)v \cos x = (1-n)\sin 2x.$

Integrating factor of this linear equation in v is

$$e^{\int (1-n)\cos x \, dx} = e^{(1-n)\sin x}$$

The solution of this equation is thus

$$ve^{(1-n)\sin x} = \int (1-n)\, e^{(1-n)\sin x} \sin 2x \, dx \qquad \qquad \text{...(1)}$$

Now, to evaluate the right hand side integral, we put $\sin x = z$, so that

$$\cos x \, dx = dz.$$

Therefore $(1-n)\displaystyle\int 2e^{(1-n)\sin x} \sin x \cos x \, dx = 2(1-n)\int e^{(1-n)z} z \, dz$

$$= 2(1-n)\left\{ z\dfrac{1}{1-n} e^{(1-n)z} - \int \dfrac{1}{1-n} e^{(1-n)z} \, dz \right\}$$

$$= 2ze^{(1-n)z} - 2\dfrac{1}{1-n} e^{(1-n)z} + c$$

$$= 2 \sin x \, e^{(1-n)\sin x} - \dfrac{2}{1-n} e^{(1-n)\sin x} + c.$$

Putting $v = y^{1-n}$ in (1), we get the general solution of the given equation as

$$y^{1-n} e^{(1-n)\sin x} = 2 \sin x \, e^{(1-n)\sin x} - \dfrac{2}{1-n} e^{(1-n)\sin x} + c$$

or, $\qquad y^{1-n} = 2 \sin x - \dfrac{2}{1-n} + ce^{(n-1)\sin x}$

Example. Solve : $\dfrac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$.

Sol. Dividing both sides of the equation by $\cos^2 y$, we get

$$\sec^2 y \, \dfrac{dy}{dx} + 2x \tan y = x^3.$$

We put tan y = v, so that $\sec^2 y \dfrac{dy}{dx} = \dfrac{dv}{dx}$ and the equation becomes

$$\dfrac{dv}{dx} + 2xv = x^3 .$$

Integrating factor of this equation is $e^{\int 2x\, dx} = e^{x^2}$.

Multiplying both sides of the equation by this integrating factor, we get

$$e^{x^2} \dfrac{dv}{dx} + 2x\, v\, e^{x^2} = x^3 e^{x^2}$$

or, $d\left(v e^{x^2}\right) = x^3 e^{x^2} dx$.

Integrating both sides, we get

$$v e^{x^2} = \int \dfrac{1}{2} z e^z \, dz, \text{ where } z = x^2$$

$$= \dfrac{1}{2}\left(z e^z - e^z\right) + c$$

or, $\tan y\, e^{x^2} = \dfrac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + c$

Hence $\tan y = \dfrac{1}{2}\ (x^2 - 1) + c e^{-x^2}$ is the general solution.

7. ORTHOGONAL AND OBLIQUE TRAJECTORIES

Orthogonal Trajectories

Let F(x, y, c) = 0 be a given one parameter family of curves in the xy plane. A curve that intersects the curves of the family at right angles is called on Orthogonal trajectory of the given family.

Procedure for finding the Orthogonal Trajectories of a given family of Curve

Step 1. From the equation F(x, y, c) = 0 ...(1)

of the given family of curves, find the differential equation $\frac{dy}{dx}$ = f(x, y) of this family.

Step 2. In the differential equation dy/dx = f(x, y)so found in Step 1, replace f(x, y) by its negative

reciprocal -1/f(x,y). This gives the differential $\frac{dy}{dx} = -\frac{1}{f(x,y)}$ equation of the orthogonal trajectories.

Step 3. Obtain a one-parameter family G(x, y, z) = 0 or y = F(x, c) of solutions of the differential equation (1). Thus obtaining the desired family of orthogonal trajectories (except possibly for certain trajectories that are vertical lines and must be determined separately).

Oblique Trajectories

Definition

Let F(x, y, c) = 0 ...(1)

be a one-parameter family of curves. A curve that intersects the curves of the family (1) at a constant angle $\alpha \neq 90°$ is called an oblique trajectory of the given family.

Suppose the differential equation of a family is $\frac{dy}{dx}$ = f(x, y) ...(2)

Then the curve of the family (2) through the point (x, y) has slope f(x, y) at (x, y) and hence its tangent line has angle of inclination tan⁻¹ [f(x, y)] there, The tangent line of an oblique trajectory that intersects this curve at the angle α will thus have angle of inclination tan⁻¹[f(x,y)] + α at the point (x, y).

Hence the slope of this oblique trajectory is given by tan {tan⁻¹f(x,y) + α} $= \frac{f(x,y) + \tan\alpha}{1 - f(x,y)\tan\alpha}$. Thus the

differential equation of such a family of oblique trajectories is given by $\frac{dy}{dx} = \frac{f(x,y) + \tan\alpha}{1 - f(x,y)\tan\alpha}$ Thus to obtain

a family of oblique trajectories intersecting a given family of curve at the constant angle $\alpha \neq 90°$, we may follow the three step in the above procedure for finding the orthogonal trajectories, except that we replace Step 2 by the following step:

Step 2. In the differential equation dy/dx = f(x, y) of the given family. Replace f(x, y) by the expression.

$$\frac{f(x,y) + \tan\alpha}{1 - f(x,y)\tan\alpha}.$$ ...(3)

Example. Consider the family of circles

$x^2 + y^2 = c^2$ ...(1)

with centre at origin and radius c. Each straight line through origin

y = kx ...(2)

is an orthogonal trajectory of the family of circles (1).

is the family of straight lines

$$y = kx \qquad \qquad ...(3)$$

Let us verify this using the procedure outlined above.

Step 1. Differentiating the equation

$$x^2 + y^2 = c^2 \qquad \qquad ...(4)$$

of the given family, we obtain

$$x + y\frac{dy}{dx} = 0$$

From this we obtain the differential equation

$$\frac{dy}{dx} = -\frac{x}{y} \qquad \qquad ...(5)$$

of the given family (4). (note that the parameter c was automatically eliminated in this case.)

Step 2. We replace $-x/y$ by its negative reciprocal y/x in the differential equation (5) to obtain the differential equation

$$\frac{dy}{dx} = \frac{y}{x} \qquad \qquad ...(6)$$

of the orthogonal trajectories.

Step 3. We now solve the differential equation (6). Separating variables, we have

$$y = kx \qquad \qquad ...(7)$$

This is a one-parameter family of solutions of the differential equation (6) and thus represents the family of orthogonal trajectories of the given family of circles (4) (except for the single trajectory that is the vertical line $x = 0$ and this may be determined by inspection).

Example. Find the orthogonal trajectories of the family of cardioids $r = a (1 - \cos \theta)$, a being a variable parameter.

Sol. From the given equation, we get by differentiating with respect to θ.

$$\frac{dr}{d\theta} = a \sin\theta \qquad \qquad ...(i)$$

Eliminating a, we get $\dfrac{dr}{d\theta} = \dfrac{r\sin\theta}{1-\cos\theta} = r\cot\dfrac{\theta}{2}$. $\qquad ...(ii)$

This is the differential equation of the given family of curves.

To get the differential equation of the system of orthogonal trajectories, we replace $\dfrac{dr}{d\theta}$ by $\left(-r^2\dfrac{d\theta}{dr}\right)$ in (ii). Thus we have

$$-r^2\frac{d\theta}{dr} = r\cot\frac{\theta}{2}, \qquad \text{that is,} \qquad \frac{dr}{r} = -\tan\frac{\theta}{2}\, d\theta.$$

Integrating both sides, we get the family of orthogonal trajectories as

$$\log r = 2 \log \cos \frac{\theta}{2} + \log (c)$$

or, $\quad r = c \cdot \cos^2 \dfrac{\theta}{2} = c (1 + \cos \theta)$.

Example. Show that the family of confocal conics

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$$

is self-orthogonal, where λ is a parameter.

Sol. Differentiating both sides of the given equation with respect to x, we get

$$\frac{x}{a^2 + \lambda} + \frac{yy_1}{b^2 + \lambda} = 0 \text{ where } y_1 = \frac{dy}{dx}.$$

This gives $\quad x(b^2 + \lambda) + yy_1(a^2 + \lambda) = 0$

or, $\quad \lambda = -\dfrac{b^2 x + a^2 yy_1}{x + yy_1}$

Therefore $\quad a^2 + \lambda = \dfrac{(a^2 - b^2)x}{x + yy_1}$ and $b^2 + \lambda = -\dfrac{(a^2 - b^2)yy_1}{x + yy_1}$

Eliminating λ from the given equation, we get the differential equation of the family of curves as

$$\frac{x^2(x + yy_1)}{(a^2 - b^2)x} - \frac{y^2(x + yy_1)}{(a^2 - b^2)yy_1} = 1$$

or, $\quad x(x + yy_1) - \dfrac{y(x + yy_1)}{y_1} = a^2 - b^2$

or, $\quad x^2 - y^2 + xy\left(y_1 - \dfrac{1}{y_1}\right) = a^2 - b^2.$ $\qquad\qquad$ ...(1)

Hence the differential equation of the family of orthogonal trajectories is

$$x^2 - y^2 + xy\left(-\frac{1}{y_1} + y_1\right) = a^2 - b^2, \qquad \left[\text{replacing } y_1 \text{ by}\left(-\frac{1}{y_1}\right)\right]$$

which is the same equation as (1) and therefore must have the same primitive.
The equation of the family of orthogonal trajectories is thus

$$\frac{x^2}{a^2 + \mu} + \frac{y^2}{b^2 + \mu} = 1$$

which represents a set of conics confocal with the given set.
Thus the system of confocal conics is self-orthogonal.

Example. Show that the orthogonal trajectories of the system of co-axial circles $x^2 + y^2 + 2\lambda x + c = 0$ forms another system of co-axial circles $x^2 + y^2 + 2\mu y - c = 0$, where λ and μ are parameters and c is a given constant.

Sol. Differentiating both sides of the given equation with respect to x, we get

$$2x + 2y\frac{dy}{dx} + 2\lambda = 0, \quad \text{giving} \quad \lambda = -\left(x + y\frac{dy}{dx}\right).$$

Eliminating λ from the given equation, we get

$$x^2 + y^2 - 2x\left(x + y\frac{dy}{dx}\right) + c = 0$$

or,
$$y^2 - x^2 - 2xy \frac{dy}{dx} + c = 0 \qquad \text{...(1)}$$

which is the differential equation of the given co-axial system of circles.

Replacing $\frac{dy}{dx}$ by $\left(-\frac{dx}{dy} \right)$, we get the differential equation of the corresponding orthogonal trajectory as

$$y^2 - x^2 + 2xy \frac{dx}{dy} + c = 0, \qquad \text{that is, } 2x \frac{dx}{dy} - \frac{x^2}{y} = -y - \frac{c}{y} \qquad \text{...(2)}$$

Let us put $x^2 = v$, so that $2x \frac{dx}{dy} = \frac{dv}{dy}$.

Then (2) becomes $\dfrac{dv}{dy} - \dfrac{1}{y} v = -y - \dfrac{c}{y}$.

This is a linear equation in v, whose integrating factor is $e^{-\log y} = \dfrac{1}{y}$.

Multiplying both sides by $\dfrac{1}{y}$ and integrating, we get

$$v \cdot \frac{1}{y} = \int \left(-y - \frac{c}{y} \right) \frac{1}{y} dy = -y + \frac{c}{y} + k,$$

where k is a constant.
Now, putting x^2 for v, we get
$$x^2 = -y^2 + c + ky$$
or, $\qquad x^2 + y^2 + 2\mu\, y - c = 0$, where $k = -2\mu$.
This is another system of co-axial circles.